9-1992

Cogrowth of Regular Graphs

Sam Northshield
SUNY Plattsburgh

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 1, September 1992

COGROWTH OF REGULAR GRAPHS

S. NORTHSHIELD

(Communicated by Lawrence F. Gray)

ABSTRACT. Let $\mathscr{G}$ be a d-regular graph and T the covering tree of $\mathscr{G}$. We define a cogrowth constant of $\mathscr{G}$ in T and express it in terms of the first eigenvalue of the Laplacian on $\mathscr{G}$. As a corollary, we show that the cogrowth constant is as large as possible if and only if the first eigenvalue of the Laplacian on $\mathscr{G}$ is zero. Grigorchuk's criterion for amenability of finitely generated groups follows.

In this note, we shall relate the first eigenvalue of the Laplacian on a connected regular graph to the size of the kernel of the universal covering map. The main results have been proven in [C, G, P]. The proof presented here appears simpler; it depends on the explicit formula for minimal positive solutions of $\Delta F + \varepsilon F = -I$.

Let $\mathscr{G}$ be a connected simple graph with constant vertex degree $d \geq 3$, T be the universal covering tree of $\mathscr{G}$, and θ the covering map (i.e., θ is a vertex surjection of T on $\mathscr{G}$ that preserves adjacency and vertex degree). We let T and $\mathscr{G}$ denote the vertex sets of the corresponding graphs. Note that T has constant vertex degree d. Since T is connected, T may be considered a metric space with the usual graph metric δ ($\delta(x, y)$ is the length of the shortest path connecting x and y). For $x \in T$ and $n \geq 0$, let $[x] = \theta^{-1}(\theta(x))$ and $S_n(x) = \{y : \delta(x, y) = n\}$. For $x, y \in T$, note that

$$\limsup_{n\to\infty} |[y] \cap S_n(x)|^{1/n} = \inf\left\{\lambda > 0 : \sum_{z\in[y]} \lambda^{-\delta(x,z)} < \infty\right\}$$

and is thus independent of x and y. We call this number, $\operatorname{cogr}(T, \mathscr{G})$, the cogrowth constant of $\mathscr{G}$ in T.

For x, y vertices of a graph, we write xEy if x and y are connected by an edge. For $x, y \in T$, let

$$q(x, y) = \begin{cases} \frac{1}{d} & \text{if } xEy, \\ 0 & \text{otherwise.} \end{cases}$$

Note that q is the transition matrix of the simple random walk on T. Let $q^{(n)}$ denote the nth power of q. For $a, b \in \mathscr{G}$ and $x \in \theta^{-1}(a)$, since θ takes the

Received by the editors January 25, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60J15; Secondary 05C05, 05C25, 43A07.
Key words and phrases. Regular graph, covering tree, amenable group, random walk.

simple random walk on T to the simple random walk on $\mathscr{G}$,

$$p^{(n)}(a, b) = \sum_{y \in \theta^{-1}(b)} q^{(n)}(x, y) \tag{1}$$

where p is the transition matrix of the simple random walk on $\mathscr{G}$. We define $\Delta \equiv p - 1$, $G \equiv \sum_{n \geq 0} p^{(n)}$, and for $\varepsilon \geq 0$, $G^{\varepsilon} \equiv \sum_{n \geq 0} p^{(n)}/(1-\varepsilon)^{n+1}$. Similarly, we define Δ_T, F, and F^{ε} as above with p replaced by q.

Δ and Δ_T are the Laplacians on $\mathscr{G}$ and T, respectively. We call $\lambda_{\mathscr{G}}$ and λ_T the first eigenvalues of Δ and Δ_T, respectively. Since T is d-regular,

$$\lambda_T = 1 - 2(d-1)^{1/2}/d \tag{2}$$

(see [DK]). Also,

$$\lambda_{\mathscr{G}} = \sup\{\lambda : \exists f > 0 : \Delta f + \lambda f \leq 0\} \tag{3}$$

(see [DK] or [N]). It is true that $\lambda_{\mathscr{G}} \leq \lambda_T$ (see [N]).

For $\varepsilon \leq \lambda_T$, let $a(\varepsilon) = d(1-\varepsilon)/(d-1)$, $b = 1/(d-1)$, and $\sigma(\varepsilon) \leq \tau(\varepsilon)$ be the (real) roots of $t = a(\varepsilon) - b/t$. Note that

$$\sigma(\varepsilon) = \{d(1-\varepsilon) - [d^2(1-\varepsilon)^2 - 4d + 4]^{1/2}\}/2(d-1),$$
$$\tau(\varepsilon) = \{d(1-\varepsilon) + [d^2(1-\varepsilon)^2 - 4d + 4]^{1/2}\}/2(d-1).$$

In particular, $\sigma(\varepsilon)$ is increasing and $\tau(\varepsilon)$ is decreasing on $[0, \lambda_T)$.

Lemma. *For* $\varepsilon \in [0, \lambda_T)$, $F^{\varepsilon}(x, y) = \sigma(\varepsilon)^{\delta(x,y)}/(1 - \varepsilon - \sigma(\varepsilon))$.

Proof. Let $\varepsilon \in [0, \lambda_T)$. For $\lambda \in (\varepsilon, \lambda_T)$, there exists a function $f > 0$ such that $\Delta_T f + \lambda f \leq 0$. Let $v = -(\Delta_T f + \varepsilon f)/(1-\varepsilon)$ and $r = q/(1-\varepsilon)$. Note that $v > 0$ and $f = v + qf$. By induction, $f = \sum_{0 \leq k \leq n} r^{(k)} v + r^{(n+1)} f \geq \sum_{0 \leq k \leq n} r^{(k)} v$ since $f > 0$. Letting $n \to \infty$, $f \geq \sum_{k \geq 0} r^{(k)} v = (1-\varepsilon) F^{\varepsilon} v$. Since $v > 0$, F^{ε} exists.

By the symmetry of T, there exists a sequence $\gamma_0, \gamma_1, \ldots$ such that for any x, if $\delta(x, y) = n$ then $F^{\varepsilon}(x, y) = \gamma_n$. Since $(\Delta_T + \varepsilon) F^{\varepsilon} = -I$, it follows that $\gamma_{k+2} = a\gamma_{k+1} - b\gamma_k$ and $\gamma_1 - (1-\varepsilon)\gamma_0 = -1$. Let $r_k = \gamma_{k+1}/\gamma_k$, $\mu = a/2$, and $\nu = [a^2/4 - b]^{1/2}$. By the addition of angle formulae for hyperbolic functions, it is easy to verify that for all r_0 there exists θ, so that

$$r_n = \begin{cases} \mu + \nu \tanh(\theta + \rho n) & \text{if } r_0 \in (\sigma, \tau), \\ \mu + \nu \coth(\theta + \rho n) & \text{if } r_0 \notin [0, \tau], \\ r_0 & \text{if } r_0 \in \{\sigma, \tau\}, \end{cases}$$

where $\rho = \tanh(\nu/\mu)$.

Clearly, if $r_0 \neq \sigma$ then $r_n \to \tau$, and thus $\lim_{n\to\infty} \gamma_n^{1/n} = \tau$. It is easy to verify that $\lim_{n\to\infty} \gamma_n^{1/n}$ is increasing as a function of ε (since $p^{(n)} \geq 0$). Therefore $r_n \equiv \sigma$ since τ is decreasing. It follows that $F^{\varepsilon}(x, y) = c\sigma^{\delta(x,y)}$ and, since $\tau_1 - (1-\varepsilon)\gamma_0 = -1$, $c = 1/[1 - \varepsilon - \sigma]$. □

Theorem. (a) $\operatorname{cogr}(T, \mathscr{G}) \leq \{d(1-\lambda_{\mathscr{G}}) + [d^2(1-\lambda_{\mathscr{G}})^2 - 4d + 4]^{1/2}\}/2$,

(b) *If* $\lambda_{\mathscr{G}} \neq \lambda_T$ *then* $\operatorname{cogr}(T, \mathscr{G}) = \{d(1-\lambda_{\mathscr{G}}) + [d^2(1-\lambda_{\mathscr{G}})^2 - 4d + 4]^{1/2}\}/2$.

Proof. (a) If $\lambda_{\mathscr{G}} = 0$ then $\operatorname{cogr}(T, \mathscr{G}) \leq \limsup_{n\to\infty} |S_n(x)|^{1/n} = d - 1 = 1/\sigma(\lambda_{\mathscr{G}})$.

Let $\lambda_{\mathscr{G}} > 0$ and $\varepsilon \in (0, \lambda_{\mathscr{G}})$. As in the proof of the lemma, G^{ε} exists. Since

$$\sum_{z\in[y]} \sigma^{\delta(x,z)} = c\sum_{z\in[y]} F^{\varepsilon}(x, z) = cG^{\varepsilon}(\theta(x), \theta(y)) < \infty,$$

$\operatorname{cogr}(T, \mathscr{G}) \leq 1/\sigma(\varepsilon)$. Since $\sigma(\varepsilon)$ is increasing, the result follows by letting ε approach $\lambda_{\mathscr{G}}$.

(b) Let $\varepsilon \in [0, \lambda_T)$. If $\operatorname{cogr}(T, \mathscr{G}) < 1/\sigma(\varepsilon)$, then

$$G^{\varepsilon}(\theta(x), \theta(y)) = \sum_{z\in[y]} F^{\varepsilon}(x, z) = \sum_{z\in[y]} \sigma^{\delta(x,z)} < \infty$$

and thus G^{ε} exists. Fix $g \in \mathscr{G}$ and let $f(x) = G^{\varepsilon}(g, x)$. Clearly $\Delta_T f + \varepsilon f \leq 0$ and $f > 0$ and, therefore, $\varepsilon \leq \lambda_{\mathscr{G}}$. Assume $\lambda_{\mathscr{G}} \neq \lambda_T$ (and thus $\lambda_{\mathscr{G}} < \lambda_T$). If $\lambda_{\mathscr{G}} < \lambda_{\mathscr{G}} + \kappa \leq \lambda_T$, then $\operatorname{cogr}(T, \mathscr{G}) \geq 1/\sigma(\lambda_{\mathscr{G}} + \kappa)$. Since $1/\sigma$ is decreasing on $[0, \lambda_{\mathscr{G}}]$, $\operatorname{cogr}(T, \mathscr{G}) \geq 1/\sigma(\lambda_{\mathscr{G}})$. □

Corollary 1. *Let $\mathscr{G}$ be connected and d-regular. Then* $\operatorname{cogr}(T, \mathscr{G}) = d - 1$ *iff* $\lambda_{\mathscr{G}} = 0$.

Let A be a finitely generated discrete group with k generators, F the free group with k generators, ϕ the canonical mapping of F onto A, and $K = \ker\theta$.

The map ϕ induces a covering map θ from T onto $\mathscr{G}$ where T and $\mathscr{G}$ are the Cayley graphs of F and A respectively. As is well known, A is amenable iff $\lambda_{\mathscr{G}} = 0$ (see [K, DK, DG]).

By [P], $\lim_{n\to\infty} |K \cap S_{2n}|^{1/2n}$ exists.

Corollary 2. *A is amenable iff* $\lim_{n\to\infty} |K \cap S_{2n}|^{1/2n} = 2k - 1$.

References

[C] J. L. Cohen, *Cogrowth and amenability of discrete groups*, J. Funct. Anal. **48** (1982), 301–309.

[DG] Y. Derriennic and Y. Guivarc'h, *Theoreme de renouvellement pour les groupes moyennables*, C. R. Acad. Sci. Paris Ser. A **277** (1973), 613–615.

[DK] J. Dodziuk and L. Karp, *Spectral and function theory of combinatorial Laplacians*, Contemp. Math., vol. 73, Amer. Math. Soc., Providence, RI, 1988, pp. 25–40.

[G] R. I. Grigorchuk, *Symmetric random walks on discrete groups*, Multi-Component Random Systems (Dobrushin and Sinai, eds.), Dekker, New York and Basel, 1980, pp. 285–325.

[K] H. Kesten, *Full Banach mean values on countble groups*, Math. Scand. **7** (1959), 146–156.

[N] S. Northshield, *Gauge and conditional gauge on negatively curved graphs*, Stochastic Anal. Appl. **9** (1991), 461–482.

[P] A. T. Paterson, *Amenability*, Math. Surveys Monographs, vol. 29, Amer. Math. Soc., Providence, RI, 1988.

Department of Mathematics, State University of New York at Plattsburgh, Plattsburgh, New York 12901

E-mail address: northssw@snyplava.bitnet