# A NOTE ON THE COMPLEXITY OF GRAPHS

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Abstract. The number of splanning trees in a finite graph is first expressed as the derivative (at 1) of a determinant and then in terms of a zeta function. This generalizes a result of Hashimoto to non-regular graphs.

Let $G$ be a finite graph. The complexity of $G$, denoted $\kappa$, is the number of spanning trees in $G$. This quantity has long been known to be related to matrices associated with $G$ (see [1]). When $G$ is regular, Hashimoto [2] expressed $\kappa$ as a limit involving the zeta function of the graph and asked if his expression still holds for irregular graphs. In this note, we show that the answer is yes. In particular, we derive a formula for the complexity as the derivative of a determinant involving the adjacency matrix of the graph and use this and a generalized version of Ihara's theorem ([3,4]) to get the desired result.

Let $G$ be a graph with $\nu$ vertices and $\varepsilon$ edges. Suppose we order the vertices: $(x_1, \ldots, x_\nu)$. The *adjacency matrix* $A = (a_{ij})$ is the matrix of zeros and ones such that $a_{ij} = 1$ if and only if $x_i$ and $x_j$ are adjacent. We let $D = (d_{ij})$ be the diagonal element with $d_{ii} = d_i - 1$ where $d_i$ is the degree of the vertex $x_i$. Finally, let $Q = D - I$. For a complex variable $u$, let

$$f(u) = det(I - uA + u^2 Q).$$

**Theorem.** *$f'(1) = 2(\epsilon - \nu)\kappa$.*

*Proof.* Let $M^u = I - uA + u^2 Q$ and let $M_k^u = (m_{k;ij}^u)$ denote the matrix $M^u$ with each entry of the $k^{th}$ row replaced by its corresponding derivative with respect to $u$. Then

$$\begin{aligned} det(M^u)' &= \sum_\pi sgn(\pi)(\prod_i m^u_{i\pi(i)})' \\ &= \sum_j \sum_\pi sgn(\pi) \prod_i m^u_{j;i\pi(i)} = \sum_j det(M^u_j). \end{aligned}$$

Note that $m^1_{k;ij} = d_i \delta_{ij} - a_{ij} + (d_k - 2)\delta_{ki}\delta_{ij}$, and thus

$$f'(1) = \sum_k det(M + (d_k - 2)R_k)$$

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where $M = M^1$ and $R_k = (r_{k;ij})$ is defined by $r_{k:ij} = \delta_{ki}\delta_{ij}$.

In general,if $\tilde{m}_{ij}$ is the cofactor of $m_{ij}$ in $M$, then

$$\begin{aligned} det(M + cR_k) &= \sum_j (M + cR_k)_{kj}(-1)^{j+k}\tilde{m}_{kj} \\ &= \sum_j m_{kj}(-1)^{j+k}\tilde{m}_{kj} + c\tilde{m}_{kk} = c\tilde{m}_{kk} \end{aligned}$$

(since $det(M) = 0$). By Biggs ([1], theorem 6.3), $\tilde{m}_{ij} = \kappa$ for all $i$ and $j$ and thus $f'(1) = \kappa \sum_k (d_k - 2) = 2(\epsilon - \nu)\kappa$. $\square$

We define the zeta function of $G$ to be

$$Z(u) = \prod_i (1 - u^{\omega_i})^{-1}$$

where $(\omega_1, \omega_2, \dots)$ are the lengths of *prime* cycles in $G$ (see [3] or [4] for definitions). A generalization of a theorem of Ihara's is that

$$Z(u) = \frac{1}{(1 - u^2)^{\epsilon - \nu} f(u)}$$

(see [3] or [4]). The following generalization of [2; theorem 7.7] is immediate.

**Corollary.** $\lim_{u \to 1^-} Z(u)(1 - u)^{\epsilon - \nu + 1} = -\frac{2^{-\epsilon + \nu - 1}}{(\epsilon - \nu)\kappa}$.

## References

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4. H.M. Stark and A.A. Terras, *Zeta functions of finite graphs and coverings*, Adv. Math. **121** (1996), no. 1, 124–165.