ON ITERATES OF MÖBIUS FUNCTIONS ON FIELDS

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Consider the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. It is known that the ratios $r_n \equiv F_{n+1}/F_n$ converge (and in fact are continued fraction convergents) to the "golden ratio" $\frac{1+\sqrt{5}}{2}$. If $m(x) = 1 + 1/x$ then, by the definition of F_n, $r_{n+1} = m(r_n)$. Associated to the sequence (r_n) is the "characteristic" polynomial $x^2 = x + 1$ whose largest root is the golden ratio. Iteration of Newton's method (with respect to this polynomial, and starting at 1), converges to this root and, in fact, gives the 1st, 2nd, 4th, 8th, 16th, etc. continued fraction convergents. Similar results hold for iteration of the secant method and for Aitken acceleration. This paper grew out of an attempt to understand these (known []) phenomena.

The function m above is an example of a Möbius function; that is, a function of the form

$$m(x) = \frac{cx + b}{x - a + c}. \tag{1}$$

Given r_0, we construct the sequence (r_n) recursively:

$$r_{n+1} = m(r_n). \tag{2}$$

Note that these definitions only require that the "numbers" be elements of a field. Henceforth, we assume we are working over a fixed but arbitrarily chosen field K.

We shall show that the relations, as mentioned above, between the iterates of m and iterates of Newton's method, the secant method, and Aitken acceleration generalize to our case. Furthermore, our proofs are different (and perhaps simpler) than the extant proofs.

Secondly, we consider convergence of these iterates. To discuss convergence, we assume that K has a topology defined by a norm (or absolute value). Furthermore, if $r_n \to \xi$ then, by (2), $\xi = m(\xi)$ and so ξ must be a zero of the characteristic polynomial

$$\theta(x) = x^2 - ax - b.$$

We shall assume then that the zeros (ξ_1 and ξ_2) of θ are in K. We shall give conditions for r_n to converge to a given root. Furthermore, we show that Newton's

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method converges quadratically, and higher order methods converge with correspondingly high degree.

Given $x, y \in K - \{\xi_1, \xi_2\}$, let

$$x \oplus y = \frac{xy + b}{x + y - a}.$$

Although it is clear that the binary relation $\oplus$ is commutative, it is perhaps less clear that it is associative. In the real case, it is a challenging problem to show *geometrically* that this is so. (The connection to geometry in this case is that the line through $(x, \theta(x))$ and $(y, \theta(y))$ has x-intercept $x \oplus y$).

Theorem 1. *The relation $\oplus$ is associative.*

Proof. Given any two-by-two matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ define an associated Möbius function $F_A(x) = \frac{ax+b}{cx+d}$. It is well known (and easy to verify) that the composition of such functions corresponds to matrix multiplication: $F_A \circ F_B = F_{AB}$.

Let $M = \begin{pmatrix} 0 & b \\ 1 & -a \end{pmatrix}$. Note that $F_{M+xI}(y) = x \oplus y$. Hence

$$(x \oplus y) \oplus z = F_{M+zI}(x \oplus y) = F_{M+zI}(F_{M+xI}(y)) = F_{(M+zI)(M+xI)}(y).$$

Since $M + zI$ and $M + xI$ commute, we have $(x \oplus y) \oplus z = (z \oplus y) \oplus x$. $\square$

We let $x^{\oplus n}$ denote the n-fold "sum" of x (i.e. $x^{\oplus 1} = x$ and $x^{\oplus(n+1)} = x \oplus x^{\oplus n}$). We now let $r_k, r_{k+1}, \ldots$ denote the iterates of m as defined above (starting at k allows for the possibility that $r_k = c$ and thus r_{k-1} is necessarily undefined. Note that, by (1),

$$m(x) = x \oplus c$$

and therefore, by a simple induction argument, for $n \geq 1$ and $k \geq 0$,

$$r_{n+k} = r_k \oplus c^{\oplus n}. \tag{3}$$

Note that by associativity and commutivity,

$$\frac{r_{m-i}r_i + b}{r_{m-i} + r_i - a} = r_{m-i} \oplus r_i = r_k \oplus r_k \oplus a^{\oplus m-k} = r_k \oplus r_{m-k}$$

is independent of i. Hence, for i and j between 1 and m,

$$\frac{r_{m-i}r_i + b}{r_{m-i} + r_i - a} = \frac{r_{m-j}r_j + b}{r_{m-j} + r_j - a}.$$

Since they are equal, they are also equal to the ratio of differences (i.e. if $A/B = C/D$, then $A/B = (A - C)/(B - D)$) and we have:

Theorem 2. *For all i, j and k such that the denominator of the fraction below is non-zero,*

$$\frac{r_{m-i}r_i - r_{m-j}r_j}{r_{m-i} + r_i - r_{m-j} - r_j} = r_k \oplus r_{m-k}.$$

This is a generalization of the Aitken acceleration formula:

Corollary 3. *If $r_1 = c$, then*

$$\frac{r_{n+m}r_{n-m} - r_n^2}{r_{n+m} - 2r_n + r_{n-m}} = r_{2n}.$$

Proof. Letting $r_1 = c$ (leaving r_0 undefined), a simple induction argument shows $r_n = c^{\oplus n}$. The result follows from letting $m = 2n, i = n - m, j = n$ and $k = 1$ in theorem 2. □

Newton's method is, given a starting point t_0, to construct a sequence

$$t_{n+1} = t_n - \frac{\theta(t_n)}{\theta'(t_n)}$$

which converges (in many cases) to a zero of θ. In our case, this boils down to

$$t_{n+1} = \frac{t_n^2 + b}{2t_n - a + c} = t_n \oplus t_n.$$

We take this to be the definition of Newton's method in the arbitrary field case. A simple induction argument shows:

Theorem 4. *If $t_0 = r_0 = c$, then $t_n = r_{2^n}$.*

One may generalize further. Let $g^{(k)}(x) = x^{\oplus k}$. Then $t_{n+1}^{(k)} = g^{(k)}(t_n^{(k)})$ defines a sequence. When $k = 3$ this gives Halley's method.

As above,

Theorem 5. *If $t_0 = r_0 = c$, then $t_n^{(k)} = r_{k^n}$.*

The secant method is, given two starting points s_0 and s_1, to construct a sequence defined by

$$s_{n+1} = s_n - \frac{\theta(s_n)(s_n - s_{n-1})}{\theta(s_n) - \theta(s_{n-1})}$$

which, in our case, boils down to

$$s_{n+1} = \frac{s_n s_{n-1} + b}{s_n + s_{n-1} - a + c} = s_n \oplus s_{n-1}.$$

As above, we take this to be the definition of the secant method in the general case.

The Fibonacci sequence shows up in a perhaps surprising way.

Theorem 6. *If $s_0 = s_1 = r_1 = c$, then $s_n = r_{F_k}$ where $(F_0, F_1, \dots)$ is the Fibonacci sequence.*

Proof. By (1), it is enough to show that $s_n = c^{\oplus n}$. For $n = 0, 1$, the result holds. Suppose it holds for $k \leq n$. Then $s_{n+1} = s_n \oplus s_{n-1} = c^{\oplus F_n} \oplus c^{\oplus F_{n-1}} = c^{\oplus(F_n + F_{n-1})} = c^{\oplus F_{n+1}}$. The result follows by induction. □

In order to discuss convergence of (r_n), we assume that there exists an "absolute value" or "norm" $|\bullet|$ on K. That is, for all $x, y \in K$,

a) $|x| = 0$ if and only if $x = 0$,

b) $|x + y| \leq |x| + |y|$, and

c) $|xy| = |x||y|$. As mentioned above, we assume that the zeros (ξ_1 and ξ_2) of θ are in K. We may then introduce a function which behaves like a "homomorphism"; let

$$f(x) = \left| \frac{x - \xi_1}{x - \xi_2} \right|.$$

Theorem 7. *For all* $x, y \in K - \{\xi_1, \xi_2\}$,

$$f(x \oplus y) = f(x)f(y).$$

Proof. Since $x^2 - ax - b = (x - \xi_1)(x - \xi_2)$, we have $\xi_1 + \xi_2 = a$ and $\xi_1\xi_2 = -b$. Hence, $z = x \oplus y = \frac{xy - \xi_1\xi_2}{x+y-\xi_1-\xi_2}$ which implies

$$\frac{z - \xi_1}{z - \xi_2} = \frac{xy - (x+y)\xi_1 + \xi_1^2}{xy - (x+y)\xi_2 + \xi_2^2}.$$

Taking the absolute value of both sides:

$$f(z) = \left| \frac{(x - \xi_1)(y - \xi_1)}{(x - \xi_2)(y - \xi_2)} \right| = f(x)f(y).$$

□

We are now able to say some things about the sequence r_n.

Theorem 8. *Let* $m(z) = z \oplus c$ *and* m_n *be the* n*-th iterate of* m.

a) If $|c - \xi_1| > |c - \xi_2|$, *then, for* $z \neq \xi_1$, $m_n(z)$ *converges to* ξ_2 *in the norm topology.*

b) If $|c - \xi_1| = |c - \xi_2|$ *but* $\xi_1 \neq \xi_2$ *then, for all* $z \notin \{\xi_1, \xi_2\}$, $m_n(z)$ *does not converge.*

c) If ξ *is the only zero of* θ *then, for all* z, $m_n(z)$ *converges to* ξ.

Proof. a) If $|c - \xi_1| > |c - \xi_2|$ then $f(c) > 1$ and, by theorem 7 and induction, $f(m_n(z)) = f(z)f(c)^n$. Unless $f(z) = 0$ (equivalently, $z = \xi_1$), $f(m_n(z)) \to \infty$. Since f is bounded outside any neighborhood of ξ_2 (triangle inequality), the result follows.

b) If $|c - \xi_1| = |c - \xi_2|$ then, $f(c) = 1$ and so, for any $z \notin \{\xi_1, \xi_2\}$, $f(m_n(z))$ is non-zero and independent of n. Since $m_n(z)$ can converge only to ξ_1 or ξ_2 in which case $f(m_n(z))$ would converge to 0 or ∞, the result follows.

c) If ξ is the only zero of θ, then $x \oplus y = \frac{xy - \xi^2}{x+y-2\xi}$ and a simple calculation gives

$$\frac{1}{x \oplus y - \xi} = \frac{1}{x - \xi} + \frac{1}{y - \xi}.$$

Hence, by induction,

$$\left| \frac{1}{m_n(x) - \xi} - \frac{1}{x - \xi} \right| = \frac{n}{|c - \xi|}$$

and the result follows. □

We say x_n converges to x with order k if $\frac{|x_{n+1} - x|}{|x_n - x|^k}$ converges to a non-zero constant. For example, Newton's method converges quadratically.

Let $t_n^{(k)}$ be defined as above.

Theorem 8. *If θ has distinct zeros and $r_n \to \xi$, then $t_n^{(k)} \to \xi$ with order k.*

Proof. Suppose $r_n \to \xi$. We write $f(x) \asymp g(x)$ if $lim_{x\to\xi} \frac{f(x)}{g(x)}$ exists and is non-zero.

Suppose y depends on x and $y \to \xi$ as $x \to \xi$. Using the fact that $\xi^2 = a\xi + b$,

$$(x-\xi)(y-\xi) = xy + b - (x+y-a)\xi = (x+y-a)(x \oplus y - \xi).$$

Since the zeros of θ are assumed distinct, $2\xi \neq \xi_1 + \xi_2 = a$ and so $|x+y-a| \asymp 1$. Hence $|x-\xi||y-\xi| \asymp |x \oplus y - \xi|$. If $|y-\xi| \asymp |x-\xi|^k$, then $|x \oplus y - \xi| \asymp |x-\xi|^{k+1}$ and so, by induction,

$$|x^{\oplus k} - \xi| \asymp |x-\xi|^k$$

and the result follows. □

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