2009

On two types of exotic addition

Sam Northshield
SUNY Plattsburgh

Recommended Citation
Northshield, S. (2009). On two types of exotic addition. *Aequationes Mathematicae, 77*, 1-23.

Revision
Feb. 6, 2008

ON TWO TYPES OF EXOTIC ADDITION

Sam Northshield

Department of Mathematics, SUNY-Plattsburgh
Plattsburgh, NY 12901

Abstract. We classify, under reasonable assumptions, all differentiable functions f for which the 'secant method' $[xf(y) - yf(x)]/[f(y) - f(x)]$ is continuous and associative. Further, we classify all differentiable functions for which the similar type of addition $xf(y) + yf(x)$ is associative.

1. INTRODUCTION.

Let $\overline{\mathbb{R}}$ be the one-point compactification of $\mathbb{R}$, $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ and consider the binary operation

$$x \oplus y = \frac{xf(y) - yf(x)}{f(y) - f(x)}. \tag{1a}$$

This operation is of course undefined when $x = y$ or when $f(x) = f(y)$. It is possible however to extend the definition of $\oplus$ for some but not all pairs of numbers x, y. We shall define a domain D upon which an exteded definition is possible and upon which $\oplus$ is closed (i.e., $x \oplus y$ is defined and in D for all $x, y \in D$) and continuous:

$$\lim_{z \to y} x \oplus z = x \oplus y.$$

This also requires certain conditions on $f(x)$. We shall list them now:

$$\textit{No line crosses the graph } y = f(x) \textit{ at more than two points.} \tag{A}$$

$$f(x) \textit{ is differentiable on } \overline{\mathbb{R}}. \tag{D}$$

In this context, differentiable shall mean that the difference quotient, for $x_0 \in \mathbb{R}$ satisfies

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists in } \overline{\mathbb{R}}$$

and at ∞ there exists a tangent line: either there exists $L \in \mathbb{R}$ such that

$$\lim_{x \to \infty} (f(x) - Lx) \text{ exists in } \mathbb{R} \tag{2a}$$

AMS 2000 Subject Classification: Primary 39B22 ; Secondary 39B12, 26A18.

Key words and phrases: Associativity, Secant Method, algebraic addition, Möbius transformation.

in which case the tangent line has equation of the form $y = Lx + b$ or

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty \tag{2b}$$

in which case we write $L = \infty$ and the tangent line is, formally, $x = \infty$. We say $f(\infty) = 0$ if $L = 0$.

We now define a domain D for $\oplus$ and define $x \oplus y$ for any $x, y \in D$: let

$$D := \overline{\mathbb{R}} - Z(f)$$

where $Z(f) := \{x : f(x) = 0\}$ is the set of zeros of f (recall $f(\infty) = 0$ if $L = 0$) and

$$x \oplus y := \begin{cases} (xf(y) - yf(x))/(f(y) - f(x)) & \text{if } x \neq y,\ f(x) \neq f(y),\ x \neq \infty \neq y \\ x - f(x)/f'(x) & \text{if } x = y,\ f'(x) \neq 0, x \neq \infty \neq y \\ x - f(x)/L & \text{if } x \neq \infty,\ y = \infty,\ L \neq \infty \\ x & \text{if } x \neq \infty,\ y = \infty,\ L = \infty \\ \infty & \text{otherwise.} \end{cases}$$

Continuity of $\oplus$ forbids more than one zero in D. To see this, suppose $z_1, z_2 \in \partial Z(f)$. Since f is continuous, $z_1, z_2 \in Z(f)$ and $z_i \oplus x = z_i$ (i=1,2) for all $x \notin Z(f)$. Hence

$$z_1 = \lim_{x\to z_2, x\notin Z(f)} z_1 \oplus x = z_1 \oplus z_2.$$

Similarly, $z_2 = z_1 \oplus z_2$ and thus $z_1 = z_2$. If there is one zero $z_0 \in D$ then either $D - \{z_0\}$ is closed under $\oplus$ or not (e.g., if $f(x) = x^2$ or $f(x) = x$ respectively). In the first case, then we can proceed with $D - \{z_0\}$ instead of D. In the second case, there exist $x_0, y_0 \in D - \{z_0\}$ such that $x_0 \oplus y_0 = z_0$. By continuity of $\oplus$, $x_0 \oplus D := \{x_0 \oplus x : x \in D\}$ is connected and contains z_0. If $f(x)$ is not linear, then $x_0 \oplus D$ contains an interval which contains z_0. Furthermore, if $\oplus$ is associative, then $y \in x_0 \oplus D$ implies $y \oplus y_0 = z_0$ and thus $(y, f(y))$ lies on the line through $(x_0, f(x_0))$ and $(y_0, f(y_0))$. It follows that $f(x)$ is linear on some interval containing z_0. We remark that this does not preclude non-trivial examples where this phenomenon could occur (the simplest such cases being where $f(x)$ is linear and the theory is trivial); rather we choose not to pursue this question at this time. It is worth noting that it is conceivable that rational functions may define an operation $\oplus$ via equation (1) on a suitably restricted domain. We do not address that question here either.

Note that condition (A) is equivalent to the *reducibility* of $\oplus$:

$$x \oplus y = x \oplus z \text{ implies } y = z.$$

At first glance, it looks that condition (A) is needlessly restrictive. Note, however, if $\oplus$ is associative and there exist three distinct points $(x, f(x))$, $(y, f(y))$, and $(z, f(z))$ all on a line, then $x \oplus y = x \oplus z = y \oplus z$. Let u be the common value. By associativity, $u \oplus x = u \oplus y$ and thus the line through $(u, f(u))$ and $(x, f(x))$ coincides with the line through $(u, f(u))$ and $(y, f(y))$ It follows that $f(u) = 0$ and thus D is not closed.

Condition (D) is necessary for $x \oplus x$ to be defined. To see this, note that $\oplus$ can be defined equivalently by

$$x \oplus y = x - \frac{f(x)}{s(x,y)} \tag{3}$$

where

$$s(x,y) := \frac{f(x) - f(y)}{x - y}$$

is the slope of the secant line through the points $(x, f(x))$ and $(y, f(y))$. Clearly, $x \oplus x$ is defined if and only if $s(x,y)$ has a limit as $y \to x$; i.e., f is differentiable and we have

$$x \oplus x = x - \frac{f(x)}{f'(x)} \text{ for } x \in \mathbb{R} - Z(f) \tag{4a}$$

with the understanding that

$$x \oplus x = \infty \text{ if } f'(x) = 0. \tag{4b}$$

Equations (2a) and (2b) help to define addition involving infinity. If (2a) holds, then

$$x \oplus \infty = x - \frac{f(x)}{L} \text{ and } \infty \oplus \infty = \lim_{x\to\infty} x \oplus \infty.$$

If (2b) holds, then

$$x \oplus \infty = x \text{ and } \infty \oplus \infty = \infty.$$

These considerations show that $x \oplus y$ is defined whenever $x \neq y$. Furthermore, $\oplus$ is closed (i.e., takes $D \times D$ to D) because if $x \neq y$ and $f(x \oplus y) = 0$, then $(x, f(x))$, $(y, f(y))$, and $(x \oplus y, 0)$ are three distinct points on a line (violating assumption (A)). A similar argument holds for $x = y$.

We claim now that $s(x,y)$ is locally monotonic in each variable; we shall show this for $y > x$ and sufficiently close to x. Suppose $x < y < z < w$ for y, z, w otherwise arbitrarily chosen from some interval on which $s(x,\cdot)$ is bounded. If $s(x,y) < s(x,z) > s(x,w)$, then let L be a line through $(x, f(x))$ with slope between $s(x,z)$ and $\max\{s(x,y), s(x,w)\}$. Then L divides the plane into two half-planes with $(z, f(z))$ in one and the points $(y, f(y))$ and $(w, f(w))$ in the other. By the intermediate value theorem, L intersects the graph at $(x, f(x))$ and two other points as well – violating condition (A). A similar argument rules out $s(x,y) > s(x,z) < s(x,w)$ and so $s(x,\cdot)$ is monotonic on some interval $(x, x+\epsilon)$.

Suppose $f(x \oplus x) = 0$. Then the tangent line through $(x, f(x))$ goes through $(x \oplus x, 0)$. Since, by (A), $f'(x)$ is locally monotonic, the curve $y = f(x)$ is locally either below or above the tangent line at $(x, f(x))$ and so a line through $(x \oplus x, 0)$ with slope slightly lower or higher than $f'(x)$ must intersect the curve at more than two places (in violation of (A)). Hence $x \oplus x \in D$.

The binary operation $\oplus$ is, as mentioned before, closely related to the *secant method* and *Newton's method*. Given two initial distinct approximations x_0 and x_1 to a root of $f(x) = 0$, we get a new approximation $x_2 := x_0 \oplus x_1$. Repeating this process to initial guesses x_1 and x_2 gives x_3 and so on. The secant method is the computation of the sequence (x_n) where

$$x_{n+1} = x_n \oplus x_{n-1}.$$

Newton's method is the iteration of the map

$$x \mapsto x \oplus x = x - \frac{f(x)}{f'(x)}.$$

For certain functions f, $\oplus$ is associative; when this is so, we say that $\oplus$ is a *secant addition* and the sequence (x_n) of approximations from the secant method becomes

$$x_n = \overbrace{x_0 \oplus \cdots \oplus x_0}^{F_{n-1} \text{ times}} \oplus \overbrace{x_1 \oplus \cdots \oplus x_1}^{F_n \text{ times}}$$

where (F_n) is the Fibonacci sequence. Furthermore, a function G satisfying

$$G(x \oplus y) = G(x) + G(y)$$

exists and can always be found and so, a closed formula for x_n can be found when G is invertible:

$$x_n = G^{-1}(F_{n-1} \cdot G(x_0) + F_n \cdot G(x_1)).$$

Several addition laws which appear in physics and mathematics are actually secant additions. For $f(x) = (c^2 - x^2)/x$, the induced secant addition is

$$x \oplus y = \frac{x + y}{1 + xy/c^2}$$

which is the law of addition of velocities in special relativity. For $f(x) = x^2$, the induced secant addition is

$$x \oplus y = \frac{1}{1/x + 1/y}$$

which is law of addition of parallel resistances. For $f(x) = 1/x$ and $f(x) = x/(1-x)$, the induced secant additions are normal addition and normal multiplication respectively.

The secant method can be thought of as a geometric way of combining two points in part of a curve to get a third point on the curve. If associative, such a method is called a "group law" and it is known that certain curves always possess a group law. For example by the Cayley-Bacharach Theorem (see, for example, the book by Silverman and Tate [ST]), cubic curves have an associative group law and, in particular, the curves

$$y^2(dx + e) = y(ax^2 + bx + c)$$

have one. For these curves, the geometric addition prescribed by the Cayley-Bacharach Theorem actually coincides with the secant method for functions of the form

$$f(x) = \frac{ax^2 + bx + c}{dx + e}. \tag{5}$$

Therefore, for all functions of the form (5), the binary operation (1) is associative. A proof of this fact appears in [N2] and uses Pascal's theorem of 1640 of which the Cayley-Bacharach theorem is a modern generalization. For completeness, we present the proof here.

Proposition 1. *If $f(x) = (ax^2 + bx + c)/(dx + e)$ where a and d are not both 0 then $\oplus$ is a secant addition.*

Proof. Pascal's theorem states that the points of intersection of opposite sides of a hexagon inscribed in a conic all lie on a common line. Specifically, If $\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5$, and $\overline{x}_6$ are six distinct points on a conic section, then the points of intersection $L(\overline{x}_1, \overline{x}_2) \cap L(\overline{x}_4, \overline{x}_5)$, $L(\overline{x}_2, \overline{x}_3) \cap L(\overline{x}_5, \overline{x}_6)$, and $L(\overline{x}_3, \overline{x}_4) \cap L(\overline{x}_6, \overline{x}_1)$ (where $L(\overline{x}, \overline{y})$ denotes the line through the points $\overline{x}$ and $\overline{y}$) all lie on a straight line.

Given real x, let $\overline{x}$ denote the point $(x, f(x))$. Given distinct real numbers x, y and z on the x-axis such that $x \oplus y \neq z$ and $x \neq y \oplus z$, consider the sequence of six points on the conic section $y = f(x)$: $\overline{x}, \overline{y}, \overline{z}, \overline{x \oplus y}, \infty$, and $\overline{y \oplus z}$. Note that the lines $L(\overline{x}, \overline{y})$ and $L(\overline{x \oplus y}, \infty)$ intersect on the x-axis at the point corresponding to $x \oplus y$. Replacing x by z, we get a similar intersection at $z \oplus y$ and so, by Pascal's theorem, the lines $L(\overline{x}, \overline{y \oplus z})$ and $L(\overline{x \oplus y}, \overline{z})$ intersect on the x-axis. By definition of $\oplus$, the first line intersects the x-axis at $x \oplus (y \oplus z)$ while the second line intersects the x-axis at $(x \oplus y) \oplus z$ and so the quantities $(x \oplus y) \oplus z$ and $x \oplus (y \oplus z)$ are equal. This shows that $\oplus$ is *weakly associative*: i.e., associativity holds for triples avoiding confluence. Associativity itself follows from differentiability (and continuity) of $f(x)$. □

As an application, we shall analyze iterates of the Möbius transformation $m(x) := (ax+b)/(cx+d)$. Given $x_0 \in \overline{\mathbb{R}}$, define (x_n) recursively by $x_{n+1} = m(x_n)$. Let $e := m^{-1}(x_0)$ and let $f(x) = \dfrac{cx^2 + (d-a)x - b}{x - e}$. Note that the numerator is a *characteristic polynomial* of m (i.e., the polynomial whose roots are the fixed points of m). Let $\oplus$ denote the secant addition induced by f. A direct calculation gives

$$x \oplus y = \frac{(a - d - ce)xy + b(x + y) - eb}{cxy - ce(x + y) + (a - d)e + b}.$$

Then $x \oplus m(e)$ is a Möbius transformation and $e \oplus x = x$ for all x. It turns out that

$$m(x) = x \oplus m(e) = x \oplus x_0.$$

This is easily seen by noting that the Möbius transformations $m(x)$ and $x \oplus m(e)$ have the same characteristic polynomials ($m(e) \oplus x = x$ iff $f(x) = 0$ iff $p(x) = 0$) and agree at some value other than a fixed point ($m(e) = e \oplus m(e)$).

By associativity, the n-fold 'sum' $c \oplus c \oplus \cdots \oplus c$ is well defined and we shall denote it by $c^{\oplus n}$. Therefore,

$$m_n(x_0) = x_0^{\oplus n}.$$

The Newton-Raphson method (or Newton's method, see [BB]) applied to the function f at x gives $x \oplus x$. Hence, the n-th iterate of the Newton-Raphson method, starting at x_0, gives $x_0^{\oplus 2^n} = m_{2^n}(x_0)$.

A further application is a short proof of 'Aitken acceleration'; see [N1]. Let x_n be a sequence defined by

$$x_{n+1} = ax_n + bx_{n-1}$$

and define

$$r_n = \frac{x_{n+1}}{x_n}.$$

Then $r_{n+1} = a + b/r_n$ and so

$$r_n = m_n(r_0),$$

the n-th iterate of the Möbius transformation $m(x) = a + b/x$. If $\oplus$ is the secant addition corresponding to the characteristic polynomial $f(x) = x^2 - ax - b$, i.e.

$$x \oplus y := \frac{xy + b}{x + y - a},$$

then $r_{n+1} = r_n \oplus a$ and therefore $r_n = r_0 \oplus a^{\oplus n}$ and, for all $n, k \geq 0$,

$$r_{n+k} = r_k \oplus a^{\oplus n}.$$

Then

$$\frac{r_n^2 + b}{2r_n - a} = r_n \oplus r_n = r_0 \oplus r_0 \oplus a^{\oplus 2n} = \frac{r_{n+k}r_{n-k} + b}{r_{n+k} + r_{n-k} - a}$$

and, since they are equal, they are also equal to the ratio of their differences:

$$r_0 \oplus r_{2n} = \frac{r_{n+k}r_{n-k} - r_n^2}{r_{n+k} - 2r_n + r_{n-k}}.$$

Other consequences of associativity appear in [N1] and [N2].

It is not true that $\oplus$ is associative for all functions f. For example, if $f(x) = x^3$, then $x \oplus y = (x^2y + xy^2)/(x^2 + xy + y^2)$ for $x \neq y$ and $x \oplus x = 2x/3$ which is not associative $[(1 \oplus 1) \oplus -1 = 2/7 \neq 0 = 1 \oplus (1 \oplus -1)]$. The question we ask now is "are there any secant additions other than those induced by functions of the form $(ax^2 + bx + c)/(dx + e)$?" The answer is, generally, 'no'.

Theorem 1. *For $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ satisfying conditions (A) and (D), $\oplus$ is associative if and only if f is of the form $f(x) = \dfrac{ax^2 + bx + c}{dx + e}$ where a and d are not both 0.*

This result contains a partial converse of Pascal's theorem in the sense that any graph of a function for which Pascal's theorem holds must be a conic section. We shall prove Theorem 1 in Section 2.

Secant addition turns out to arise via matrices. Suppose A is a two-by-two matrix. Since A^2 is a linear combination of A and I (by the Cayley-Hamilton theorem), then either, for some c,

$$(A - xI)(A - yI) = cI$$

or there exist unique numbers k and z such that

$$(A - xI)(A - yI) = k(A - zI).$$

To see this, suppose $A^2 = aA + bI$. If $x + y = a$, then the first case holds with $c = xy + b$ while if $x + y \neq a$, then the second case holds with $k = a - x - y$ and $z = (xy + b)/(x + y - a)$.

Then z depends on x and y and we shall define a type of addition $\oplus$ by $x \oplus y := z$. We let $\doteq$ denote *projective equality*: $A \doteq B$ if and only if $A = kB$ for some scalar k, and thus

$$(A - xI)(A - yI) \doteq A - (x \oplus y)I.$$

The operation $\oplus$ inherits associativity from matrix multiplication:

$$A - (x \oplus (y \oplus z))I \doteq (A - xI)(A - yI)(A - zI) \doteq A - ((x \oplus y) \oplus z)I.$$

It will be shown (Theorem 2) that $\oplus$ is secant addition induced by the characteristic polynomial f of A and, in fact, all secant additions can be interpreted in this way.

Theorem 2. *Suppose $\oplus$ is an associative, and commutative binary operation. The following are equivalent:*

a) $\oplus$ is of secant type, i.e.,

$$x \oplus y = \frac{xf(y) - yf(x)}{f(y) - f(x)}$$

for some function f satisfying conditions (A) and (D)

b) $\oplus$ is of secant type with respect to a function f of the form

$$f(x) = \frac{ax^2 + bx + c}{dx - e}$$

where a and d are not both 0,

c) $\dfrac{\partial}{\partial x}(x \oplus y) = \dfrac{p(x \oplus y)}{p(x)}$ for some polynomial p of degree at most 2,

d) there exists $e \in \overline{\mathbb{R}}$ such that $x \oplus e = x$ and, for all $c \in \mathbb{R}$ the function $m(x) := x \oplus c$ is either a Möbius transformation or is constant,

e) there exists a two-by-two real matrix A and $e \in \overline{\mathbb{R}}$ not in the spectrum of A such that

$$(A - xI)(A - yI) \doteq (A - (x \oplus y)I)(A - eI)$$

where $M \doteq N$ means $M = cN$ for some $c \in \mathbb{R}$ (and $A - \infty I = -I$).

Secant addition also arises in the theory of differential equations. For example, $y := \cos x$ is a solution of $y'' + y = 0$ and $-y'/y$ $(= \tan x)$ satisfies

$$\tan(x + y) = \tan(x) \oplus \tan(y)$$

where

$$u \oplus v := \frac{u + v}{1 - uv}$$

is the secant addition induced by the characteristic function $x^2 + 1$. This can be broadly generalized.

Theorem 3. *If y is a solution of $y'' - \alpha y' + \beta y = 0$ and $T(x) := -y'(x)/y(x)$ then T satisfies the functional equation*

$$T(x+y) = T(x) \oplus T(y)$$

where $\oplus$ is the secant addition (1) generated by the function

$$f(x) = \frac{x^2 + \alpha x + \beta}{x - T(0)}$$

if $T(0) \neq \infty$ (equivalently, $y(0) \neq 0$) or generated by

$$f(x) = x^2 + \alpha x + \beta$$

if $T(0) = \infty$ (equivalently, $y(0) = 0$).

Hence, by Theorem 1, all secant additions arise in this way.

We next consider another class of binary operations. For example, consider

$$x \diamond y := x\sqrt{1-y^2} + y\sqrt{1-x^2} \text{ for } x, y \in [-1, 1].$$

It turns out to be associative since the sine function acts as a homomorphism:

$$\sin(x+y) = \sin(x) \diamond \sin(y) \text{ for } x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

For this reason, we say that any *associative* binary operation defined by

$$x \diamond y = x\varphi(y) + y\varphi(x) \tag{6}$$

for some differentiable function φ is a *sine-type addition.*

It turns out that sine-type additions always arise from differential equations.

Corollary 1. *If $y'' - \alpha y' + \beta y = 0$, $y(0) = 0$, and $y'(0) = 1$, and if $\varphi \circ y = y' - \frac{1}{2}\alpha y$, then the addition defined by (6) is associative. Furthermore, every sine-type addition arises this way.*

This is a corollary of the following theorem:

Theorem 4. *Suppose $\diamond$ is a binary operation of the form*

$$x \diamond y = x\varphi(y) + y\varphi(x) \text{ for } x, y \in I$$

where I is an interval containing 0, φ is differentiable, $a := \varphi'(0)$, $\varphi(0) = 1$, and

$$s^{-1}(x) = \int_0^x \frac{dt}{\varphi(t) + at}.$$

The following are equivalent:
a) $\diamond$ *is associative,*
b) $\varphi(x \diamond y) = \varphi(x)\varphi(y) + kxy$ *for some* $k \in \mathbb{R}$,
c) $\varphi'(x \diamond y) = \dfrac{\varphi'(x)\varphi(y) + ky}{\varphi(y) + y\varphi'(x)}$,
d) $\varphi'(x) = \dfrac{a\varphi(x) + kx}{\varphi(x) + ax}$,
e) $s'' - 2as' + (a^2 - k)s = 0,\ s(0) = 0,\ s'(0) = 1$,
f)

$$s(x) = \begin{cases} \frac{1}{\sqrt{k}} e^{ax} \sinh(\sqrt{k}x) & \text{if } k > 0, \\ \frac{1}{\sqrt{-k}} e^{ax} \sin(\sqrt{-k}x) & \text{if } k < 0, \\ xe^{ax} & \text{if } k = 0. \end{cases}$$

g) $s(x + y) = s(x) \diamond s(y)$,
h) $s^{-1}(x \diamond y) = s^{-1}(x) + s^{-1}(y)$,
i) $\dfrac{\partial}{\partial x}(x \diamond y) = \dfrac{\varphi(x \diamond y) + a \cdot (x \diamond y)}{\varphi(x) + ax}$.

Suppose $\diamond$ is a sine-type addition induced by the function φ and let $\Phi(x) := \varphi(x)/x$. By the equivalence of (a) and (b) in Theorem 4,

$$\Phi(x \diamond y) = \Phi(x) \oplus \Phi(y)$$

where $\oplus$ is the secant addition induced by $f(x) = x^2 - k$:

$$x \oplus y = \frac{xy + k}{x + y}.$$

The solution of the equation $g(x) + g(y) = h(xf(y) + yf(x))$ for all three unknown functions f, g and h was accomplished by Abel in 1827 and uses the idea of reducing to differential equations. Cayley also considered this equation; see [A1, p.8] for references. For continuous solutions (i.e., partial solution of the second part of Hilbert's fifth problem), see the several papers by Sablik [S1,S2] and a paper by Aczel [A2]. See also the comments after Lemma 4 below.

The theory covered in this paper may also be done over the Riemann sphere $\overline{\mathbb{C}}$; several of our arguments, however, depend on the order properties of the real line and do not carry over *verbatim* to the complex case.

Another natural setting for this theory is projective space (either real or complex). To some extent, this type of thing has been done; we now present some connections between our work and known results. Define $\varphi : D \to \mathbb{R}^2$ by

$$\varphi(x) = \left(\frac{1}{f(x)}, \frac{x}{f(x)} \right)$$

with special cases $\varphi(\infty) := (0, 1/L)$ if $\lim_{x\to\infty} f(x)/x = L$ and $\varphi(e) = (0, 0)$ if $f(e) = \infty$.

Let $\mathcal{C} := \varphi(D)$. Note that $\mathcal{C}$ inherits property (A); that is, no three points $\varphi(x), \varphi(y), \varphi(z)$ lie on a straight line for, if not, there exist numbers x, y, z, and t such that

$$(1-t)\left(\frac{1}{f(x)}, \frac{x}{f(x)}\right) + t\left(\frac{1}{f(y)}, \frac{y}{f(y)}\right) = \left(\frac{1}{f(z)}, \frac{z}{f(z)}\right)$$

which, for $a := (1-t)f(z)/f(x)$ and $b := tf(z)/f(y)$, implies

$$a(x, f(x)) + b(y, f(y)) = (z, f(z)).$$

By Lemma 1, $\mathbf{0} := (0,0) \in \mathcal{C}$. We introduce an equivalence relation on $\mathbb{R}^2$:

$$(x,y) \doteq (u,v) \text{ if there exists } k \in \mathbb{R} - \{0\} \text{ such that } x = ku \text{ and } y = kv.$$

We may then define a binary operation '$*$' on $\mathcal{C}$: given $u, v \in \mathcal{C}$, let $u * v$ be the unique element in $\mathcal{C}$ such that $u * v \doteq u - v$. Geometrically, $u * v$ is that point on $\mathcal{C}$ such that the line through it and $\mathbf{0}$ is parallel to the line through u and v.

Proposition 2. *For $x, y \in D$,*

$$\varphi(x) * \varphi(y) = \varphi(x \oplus y).$$

Proof. $\varphi(x \oplus y) \doteq (1, x \oplus y) \doteq (f(y) - f(x), xf(y) - yf(x))$

$$\doteq \left(\frac{1}{f(x)} - \frac{1}{f(y)}, \frac{x}{f(x)} - \frac{y}{f(y)}\right) = \varphi(x) - \varphi(y) \doteq \varphi(x) * \varphi(y). \square$$

Hence $*$ and $\oplus$ are conjugate. In particular, the associativity of $\oplus$ is equivalent the associativity of $*$. We note other properties equivalent to the associativity of $*$. We define a binary relation $\sim$ on $\mathcal{C} \times \mathcal{C}$:

$$(u,v) \sim (w,z) \text{ iff } z - u \doteq w - v \text{ iff } z * u = w * v.$$

This equivalence relation appears (we correct a misprint) in problem 21 of the ISFE #9 meeting website [Au] where it is stated that this is equivalent to $\mathcal{C}$ being a conic (under certain additional conditions).

Given two points A, B in the plane, let $\overline{AB}$ denote the line through A and B and, for lines L_1 and L_2, let $L_1 || L_2$ mean that L_1 and L_2 are parallel. We now define the *Parallel Hexagon Property* (PHP) for a curve $\mathcal{C}$: if A, B, C, D, E, F are six points in $\mathcal{C}$, and if $\overline{AB} || \overline{DE}$ and $\overline{BC} || \overline{EF}$ then $\overline{CD} || \overline{FA}$. This property is possessed by all conics in that it is a special case of Pascal's theorem (i.e., two pairs of opposite sides of an inscribed hexagon cross at infinity implies that the third pair does as well – see Proposition 1).

Proposition 3. *The transitivity of $\sim$ is equivalent to the associativity of $*$ which is equivalent to the PHP.*

Proof. Assume the associativity of $*$ and suppose $(a,b) \sim (u,v)$ and $(u,v) \sim (c,d)$. That is, $a * v = b * u$ and $u * d = v * c$. Then $a * v * d = b * u * d = b * v * c$

and, by the reducibility of $*$, $a * d = b * c$ and thus $(a, b) \sim (c, d)$. Therefore $\sim$ is transitive.

Suppose now that $\sim$ is transitive. Let $A, B, C, D, E, F \in \mathcal{C}$ where $\overline{AB}||\overline{DE}$ and $\overline{BC}||\overline{EF}$. The first shows that $A - B \doteq D - E$ and thus $(A, D) \sim (B, E)$ and, similarly, the second shows that $(B, E) \sim (C, F)$. By transitivity, $(A, D) \sim (C, F)$ and thus $\overline{CD}||\overline{FA}$ and the PHP holds.

Supposing the PHP, let $u, v, w \in \mathcal{C}$ and consider the sequence $\mathbf{0}, v*w, u, v, w, u*v$. Since, by definition $\overline{\mathbf{0}(u * v)}||\overline{uv}$ and $\overline{\mathbf{0}(v * w)}||\overline{vw}$, we have by the PHP,

$$\overline{u(v * w)}||\overline{(u * v)w}.$$

That is, $u * (v * w) = (u * v) * w$ and thus $*$ is associative. $\square$.

We sketch why associativity of $\oplus$ (and thus of $*$) implies that $\mathcal{C}$ is a conic. Let $\mathcal{C}_0$ be a segment of the curve $\mathcal{C}$ which is the graph of some function g over some interval I. We may define the *g-differential mean* value on $\mathcal{C}_0$ (cf [K], [Bo], [A3])

$$D_g(x, y) := (g')^{-1}\left(\frac{g(x) - g(y)}{x - y}\right).$$

Aumann showed (see [K], [A3] and references therein) that $\mathcal{C}_0$ is part of a conic section if D_g is a quasi-arithmetic mean: i.e. if there exists a function F such that

$$D_g(x, y) = F^{-1}\left(\frac{F(x) + F(y)}{2}\right).$$

By [A1, p. 281], this occurs if D_g is bisymmetric; if we define '$\cdot$' on $\mathcal{C}_0$ by $u \cdot v$ being the unique point in $\mathcal{C}_0$ with tangent line parallel to the line through u and v, then bisymmetry of D_g translates to the bisymmetry of $\cdot$:

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d).$$

This follows from the associativity of *$*$ since

$$a * b = (a \cdot b) * (a \cdot b)$$

and thus

$$((a \cdot b) \cdot (c \cdot d))^{*4} = ((a \cdot b) * (c \cdot d))^{*2} = a * b * c * d.$$

We say that *Pascal's Property* holds if the conclusion of Pascal's theorem holds (see Proposition 1).

Proposition 4. *The following are equivalent: Pascal's Property holds, the Parallel Hexagon Property holds, $\oplus$ is associative, * is associative, $\sim$ is an equivalence relation, $\cdot$ is bisymmetric, D_g is quasi-arithmetic, $\mathcal{C}$ is a conic section.*

The operation $\oplus$ is, under our assumptions, a group law. It turns out that all rational formal group laws have been classified by Coleman and McGuinness [CMc].

The author thanks the referees for many suggestions and references.

2. PROOF OF THEOREM 1.

Let $f(x)$ satisfy conditions (A) and (D) and suppose $\oplus$ is associative. In view of Proposition 1, to prove Theorem 1 it is enough to show that $f(x)$ is of the form $(ax^2+bx+c)/(dx+e)$ where a and d are not both 0.

Recall $s(x,y) := (f(x)-f(y))/(x-y)$ is the slope of the secant line through the points on the graph of f corresponding to x and y.

Lemma 1. *$s(x,y)$ is unbounded. Consequently, $f(e)=\infty$ for some (unique) $e \in \mathbb{R}$ or $\lim_{x\to\infty} f(x)/x = \infty$.*

Proof. Suppose that $s(x,y)$ is bounded. Equation (2b) is ruled out and so (2a) holds: for some $L \in \mathbb{R}$,

$$\lim_{x\to\infty}(f(x)-Lx) \text{ exists in } \mathbb{R}.$$

Fix x_0. As $x \to \infty$ (from both positive and negative directions),

$$s(x_0,x) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-Lx+L(x-x_0)+Lx_0-f(x_0)}{x-x_0} \to L.$$

Note that $s(x_0,\cdot)$ is continuous with a removable singularity at x_0 (in particular, $s(x_0,x_0)=f'(x_0)$). If $f'(x_0)\neq L$ then any line through $(x_0,f(x_0))$ with slope between L and $f'(x_0)$ must intersect the graph of $f(x)$ at three or more points (contradicting (A)). Hence $f'(x)$ is constant, again contradicting (A). $\square$

By Lemma 1, we may define a function:

$$p(x) := \begin{cases} f(x), & \text{if } \lim_{x\to\infty} f(x)/x = \infty \\ f(x)(x-e), & \text{if } f(e)=\infty. \end{cases}$$

Lemma 2. $\dfrac{\partial}{\partial x}(x\oplus y) = \dfrac{p(x\oplus y)}{p(x)}$.

Proof. Note $s(x,y) = \dfrac{f(x)}{x - x\oplus y}$. Let e' be e if $f(e)=\infty$, otherwise let $e'=\infty$. In either case, $\lim_{z\to e'} x\oplus z = x$. Then, by associativity, $x\oplus y\oplus z$ is well defined and

$$\frac{\partial}{\partial x}(x\oplus y) = \lim_{z\to e'}\frac{x\oplus y - x\oplus y\oplus z}{x - x\oplus z} = \lim_{z\to e'}\frac{f(x\oplus y)}{s(x\oplus y,z)}\frac{s(x,z)}{f(x)} = \frac{f(x\oplus y)}{f(x)}\lim_{z\to e'}\frac{s(x,z)}{s(x\oplus y,z)}.$$

Note that

$$\lim_{z\to e'}\frac{s(x,z)}{s(x\oplus y,z)} = \lim_{z\to e'}\frac{f(x)-f(z)}{x-z}\,\frac{x\oplus y - z}{f(x\oplus y)-f(z)} = \lim_{z\to e'}\frac{x\oplus y - z}{x-z}$$

which equals 1 if $e' = \infty$ and equals $\dfrac{x \oplus y - e}{x - e}$ if $e' = e$ □

The following Lemma is a well-known characterization of polynomials of degree at most two. It perhaps goes back to Lagrange.

Lemma 3. *If for all x, y $(x \neq y)$ in an interval,*

$$\frac{g'(x) + g'(y)}{2} = \frac{g(x) - g(y)}{x - y},$$

then g is a polynomial of degree at most two.

Proof. Observe that the hypothesis implies that $g(x)$ is twice differentiable: fix $y \neq x$ and note that since the right side is differentiable at x, then so is the left. Equating the partial derivatives (with respect to x) of each side, we find

$$g''(x)/2 = \frac{g'(x) - (g(x) - g(y))/(x - y)}{x - y} = \frac{g'(x) - g'(y)}{2(x - y)}$$

and thus $g''(x) = g''(y)$. Hence g'' is constant. □

Proof of Theorem 1. It is easy to verify that

$$\frac{\partial}{\partial x} s(x, y) = \frac{f'(x) - s(x, y)}{x - y}.$$

Since, by (3),

$$x \oplus y = y - \frac{f(y)}{s(x, y)},$$

we find

$$\frac{\partial}{\partial x}(x \oplus y) = \frac{f(y)}{s(x, y)^2} \cdot \frac{f'(x) - s(x, y)}{x - y}. \tag{7}$$

By Lemma 2, there are two cases. If

$$\frac{\partial}{\partial x}(x \oplus y) = \frac{f(x \oplus y)}{f(x)},$$

then, using equation (7), we may write

$$\frac{f(x \oplus y)}{f(x) f(y)} s(x, y)^2 = \frac{f'(x) - s(x, y)}{x - y}.$$

Since the left side is symmetric in x and y, so is the right side and we find

$$f'(x) - s(x, y) = s(x, y) - f'(y)$$

or, equivalently,

$$\frac{f'(x) + f'(y)}{2} = \frac{f(x) - f(y)}{x - y}.$$

By Lemma 3, $f(x) = ax^2 + bx + c$ for some a, b, c.
The other case is when

$$\frac{\partial}{\partial x}(x \oplus y) = \frac{f(x \oplus y)(x \oplus y - e)}{f(x)(x - e)}.$$

Using equation (7) here, we may write

$$\frac{f(x \oplus y)(x \oplus y - e)}{f(x)f(y)} s(x,y)^2 = \frac{(x - e)(f'(x) - s(x,y))}{x - y}.$$

Since the left side is symmetric in x and y, so is the right side and we find

$$(x - e)(f'(x) - s(x,y)) = (y - e)(s(x,y) - f'(y))$$

or, equivalently,

$$\frac{(x - e)f'(x) + (y - e)f'(y)}{x + y - 2e} = \frac{f(x) - f(y)}{x - y}.$$

Letting $q(x) := xf(x + e)$, $u = x - e$, and $v = y - e$, we may rewrite this as

$$\frac{q'(u) + q'(v)}{2} = \frac{q(u) - q(v)}{u - v},$$

and therefore, by Lemma 3, $q(u) = au^2 + bu + c$ for some a, b, c. Then

$$f(x) = \frac{q(x - e)}{x - e} = \frac{ax^2 + (b - 2ae)x + (ae^2 - be + c)}{x - e}. \quad \square$$

3. PROOF OF THEOREMS 2 AND 3.

Proof of Theorem 2. ($a \iff b$): This is Theorem 1.

($b \Rrightarrow c$): If (b) holds, then $s(x, y)$ is unbounded. By Lemma 2, (c) holds.

($c \Rrightarrow d$): The Schwarzian derivative

$$S(m) = (m''/m')' - \frac{1}{2}(m''/m')^2$$

is zero if and only if m is a Möbius transformation. Let $m(x) = x \oplus k$. By Lemma 2, since $m'(x) = \frac{\partial}{\partial x} x \oplus k = p(x \oplus k)/p(x)$,

$$m' = \frac{p \circ m}{p}.$$

Using this to work out its Schwarzian derivative, $S(m) = (q \circ m - q)/p^2$ where $q = pp'' - \frac{1}{2}(p')^2$. Since p is a polynomial of degree at most 2, q is constant and so $S(m) = 0$.

($d \Rrightarrow e$): Given a matrix $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define the corresponding Möbius function

$$\Phi_M(x) := \frac{ax+b}{cx+d}.$$

By (d), there exist matrices $A(x)$ such that

$$x \oplus y = \Phi_{A(x)}(y).$$

Hence, for $x, y, z \in \mathbb{R}$,

$$\Phi_{A(x)A(z)}(y) = \Phi_{A(x)}(\Phi_{A(z)}(y)) = x \oplus (y \oplus z)$$

and thus the matrices $A(x)$ all commute (modulo the equivalence relation $\doteq$). Furthermore, for all x, y,

$$A(x \oplus y) \doteq A(x)A(y).$$

Fix $k \in \mathbb{R}$ such that

$$B := \begin{pmatrix} a & b \\ c & d \end{pmatrix} := A(k)$$

is invertible and not a multiple of the identity. Then, without loss of generality, there is a function γ such that for all x

$$A(x) = A(k) - \gamma(x)I.$$

Writing $x \oplus k$ two ways, we find

$$\frac{k\gamma(x) - (ka+b)}{\gamma(x) - (kc+d)} = \frac{ax+b}{cx+d}$$

and thus γ is a Möbius transformation, say

$$\gamma(x) = \frac{sx+t}{ux+v}.$$

Since $A(x)$ is invertible if and only if $\gamma(x) \notin Sp(B)$, $uB - sI \doteq A(\infty)$ is invertible. Let $C = (vB + tI)(uB - sI)^{-1}$. Then

$$A(x) \doteq (ux+v)B - (sx+t)I = (C - xI)(uB - sI).$$

Note that $A(e) \doteq I$ and thus

$$C - eI \doteq (uB - sI)^{-1}.$$

Then

$$\begin{aligned}(C - xI)(C - yI) &\doteq A(x)A(y)(uB - sI)^{-2} \\ &\doteq A(x \oplus y)(uB - sI)^{-2} \\ &\doteq (C - (x \oplus y)I)(uB - sI)^{-1} \\ &\doteq (C - (x \oplus y)I)(C - eI).\end{aligned}$$

($e \Rrightarrow b$): We have two main cases. Suppose first that A is not a multiple of I. Let A have trace T and determinant Δ. Then, by the Cayley-Hamilton theorem,

$$A^2 = TA - \Delta I,$$

and so

$$(A - xI)(A - yI) = (T - x - y)A + (xy - \Delta)I.$$

By (e), if $z := x \oplus y$, then

$$(T - z - e)A + (ze - \Delta)I \doteq (T - x - y)A + (xy - \Delta)I$$

and, since A is not a multiple of I,

$$\frac{ze - \Delta}{z + e - T} = \frac{xy - \Delta}{x + y - T}.$$

Solving for $z(=x \oplus y)$, we get:

$$x \oplus y = \frac{(T - e)xy - \Delta(x + y) + e\Delta}{xy - e(x + y) + eT - \Delta}.$$

A direct calculation shows that if $f(x) = (x^2 - Tx + \Delta)/(x - e)$, then (b) holds for the function f.

Now, suppose $A = cI$ for some $c \in \mathbb{R}$. By (e), if $z := x \oplus y$, then

$$(c - x)(c - y) = k(c - z)(c - e)$$

for some k. Solving for $z(=x \oplus y)$, we get

$$x \oplus y = \frac{xy - c(x + y) + c^2 - kc^2 + kce}{k(e - c)}.$$

A direct calculation shows that if $f(x) = (x - c)/(x - c + k(c - e))$, then (b) holds for the function f. □

Proof of Theorem 3. Let $y'' - \alpha y' + \beta y = 0$ and $T := -y'/y$. If we define $z := -y'$, then

$$\begin{cases} z' = \alpha z + \beta y \\ y' = -z. \end{cases}$$

By the quotient rule, $(z/y)' = (z/y)^2 + \alpha(z/y) + \beta$ and thus

$$T' = p(T)$$

where $p(x) = x^2 + \alpha x + \beta$. This is a separable equation and so, if $F := T^{-1}$, then $F' = 1/p$ and $F(T(0)) = 0$.

Let $f(x) = p(x)/(x - T(0))$ or $p(x)$ according to whether $T(0)$ is finite or not. By Lemma 2,

$$\frac{\partial}{\partial x} F(x \oplus y) = F'(x \oplus y) \frac{\partial}{\partial x}(x \oplus y) = \frac{1}{p(x \oplus y)} \cdot \frac{p(x \oplus y)}{p(x)} = \frac{1}{p(x)} = F'(x)$$

and thus $F(x \oplus y) = F(x) + G(y)$ for some function G. By commutativity, $F(x) + G(y) = F(y) + G(x)$ from which it follows that $F(x) - G(x) = F(y) - G(y)$, and so F and G differ by a constant (say c). Hence,

$$F(x \oplus y) = F(x) + F(y) + c.$$

The condition $F(T(0)) = 0$ ensures that this constant is zero. □

4. ON SINE TYPE ADDITION.

We say that the binary operation $\diamond$ is of *sine-type* if it is of the form

$$u \diamond v = u\varphi(v) + v\varphi(u)$$

for some function φ and is associative.

All examples of sine-type addition arise from secant type addition (Corollary 1). As an example, let $f(x) = 1/x$. The secant addition induced by f is normal addition

$$x \oplus y = x + y$$

and thus $T(x) := x$ is a homomorphism:

$$T(x + y) = T(x) \oplus T(y).$$

Suppose that there are functions S and C such that $T(x) = S(x)/C(x)$. Then

$$\frac{S(x+y)}{C(x+y)} = \frac{S(x)C(y) + C(x)S(y)}{C(x)C(y)}$$

and it is not unreasonable that S and C satisfy the system of functional equations

$$\begin{cases} S(x+y) = & S(x)C(y) + C(x)S(y) \\ C(x+y) = & C(x)C(y). \end{cases}$$

This system is solvable: $C(x) = e^{kx}$, and thus $S(x) = xe^{kx}$. Since $u \mapsto (u \ln u)/k$ takes C to S, if $\varphi(x)$ is the inverse of $x \ln x$, then

$$S(x + y) = S(x)\varphi(S(y)) + S(y)\varphi(S(x)).$$

That is, if

$$u \diamond v = u\varphi(v) + v\varphi(u)$$

is the sine-type addition defined by φ, then

$$S(x + y) = S(x) \diamond S(y).$$

Lemma 4. *Let $\diamond$ be the binary operation defined by*

$$x \diamond y = x\varphi(y) + y\varphi(x)$$

on some interval I containing 0 for some function φ. Then $\diamond$ is associative if and only if, for some $k \in \mathbb{R}$,

$$\varphi(x \diamond y) = \varphi(x)\varphi(y) + kxy.$$

Proof. Suppose $\diamond$ is associative. Then

$$x\varphi(y)\varphi(z) + z\varphi(x \diamond y) = x \diamond y \diamond z - y\varphi(x)\varphi(z) = z\varphi(y)\varphi(x) + x\varphi(z \diamond y),$$

and therefore

$$\frac{\varphi(z \diamond y) - \varphi(z)\varphi(y)}{z} = \frac{\varphi(x \diamond y) - \varphi(x)\varphi(y)}{x}$$

is dependent on y only and so, if we call this quantity $g(y)$, then, solving for $\varphi(x \diamond y)$,

$$\varphi(x \diamond y) = \varphi(x)\varphi(y) + xg(y).$$

By commutativity, $xg(y) = yg(x)$ and so $g(y) = ky$ for some k. It follows that

$$\varphi(x \diamond y) = \varphi(x)\varphi(y) + kxy.$$

Conversely, suppose $\varphi(x \diamond y) = \varphi(x)\varphi(y) + kxy$. Then

$$(x \diamond y) \diamond z = z\varphi(x)\varphi(y) + kxyz + x\varphi(y)\varphi(z) + y\varphi(x)\varphi(z)$$

which, since is symmetric with respect to x and z, implies associativity. □

The equation

$$\varphi(x \diamond y) = \varphi(x)\varphi(y) + kxy \tag{8}$$

has been much studied. For $k = 0$, (8) has been studied by Brillouet and Dhombres in [BD]. Sablik [S2] has noted that, at least for $k > 0$, the substitution $F(x) := \varphi(x/2\sqrt{k}) + x/2$ allows one to rewrite (8) as the 'Baxter equation':

$$F(xF(y) + yF(x) - xy) = F(x)F(y). \tag{9}$$

This has been studied by Benz [Be] where it is shown that there are infinitely many discontinuous solutions and where a geometric problem related to (9) is presented. The (unique) continuous solution of (9) has been determined by Volkman and Weigel [VW] and, later, with a shorter proof, by Brzdek [Br].

Proof of Theorem 4. Lemma 4 shows the equivalence of (a) and (b).

($b \Rrightarrow c$): Since $(x \diamond y)' = \varphi(y) + y\varphi'(x)$, differentiating and equating both sides of (b) and then solving for $\varphi'(x \diamond y)$ gives (c).

($c \Rrightarrow d$): Letting $x = 0$ in (c) and using $\varphi'(0) = a$ gives (d).

($d \Rrightarrow e$): By the definition of $s^{-1}(x)$, $s'(s^{-1}(x)) = \varphi(x) + ax$ and so

$$\varphi(s(x)) = s'(x) - as(x), \qquad s(0) = 0, \qquad s'(0) = 1. \tag{8}$$

Using (d) and (8),

$$s''(x) - as'(x) = \varphi'(s(x))s'(x) = a\varphi(s(x)) + ks(x) = a(s'(x) - as(x)) + ks(x)$$

and (e) follows.

($e \Rrightarrow f$): Solving the equation, (f) follows.

($f \Rrightarrow g$): Assuming (f), a direct calculation shows

$$s(x + y) = s(x)s'(y) + s(y)s'(x) - 2as(x)s(y).$$

Using equation (8), (g) follows.

($g \Rrightarrow h$): By definition.

($h \Rrightarrow i$): Partial differentiation of both sides of (h) yields

$$(s^{-1})'(x \diamond y)\frac{\partial}{\partial x}(x \diamond y) = (s^{-1})'(x).$$

By the integral definition of s^{-1}, (i) follows.

($i \Rrightarrow b$): By (i),

$$\varphi(y) + y\varphi'(x) = \frac{\varphi(x \diamond y) + a \cdot (x \diamond y)}{\varphi(x) + ax}$$

which, solving for $\varphi(x \diamond y)$, yields

$$\varphi(x \diamond y) = \varphi(x)\varphi(y) + y[\varphi(x)\varphi'(x) + ax\varphi'(x) - a\varphi(x)].$$

By commutativity, (b) follows. □

We indicate why Corollary 1 follows from Theorem 4. Note that the equation

$$\varphi \circ y = y' - \frac{1}{2}\alpha y$$

can be written in integral form:

$$y^{-1}(x) = \int_0^x \frac{dt}{\varphi(t) - \frac{1}{2}\alpha t}.$$

This coincides with the definition of s in Theorem 4 and so Corollary 1 follows by the equivalence of conditions (a) and (e) in Theorem 4.

5. CONCLUSION.

We have discussed two types of algebraic addition. In both cases, there exists a 'homomorphism'. For example, sine-type addition with $\varphi(x) := \sqrt{1 - x^2}$ satisfies

$$\sin(x + y) = \sin(x) \diamond \sin(y)$$

and secant addition with $f(x) := (1 - x^2)/x$ satisfies

$$\tanh(x + y) = \tanh(x) \oplus \tanh(y).$$

In general, there exists F such that

$$F(x+y) = F(x) * F(y).$$

Furthermore, the functions F are invertible and, for fixed c,

$$F^{-1}(x * c) = F^{-1}(x) + F^{-1}(c).$$

If $g(x) := F^{-1}(x)/F^{-1}(c)$ and $m(x) := x*c$, then g and m satisfy 'Abel's equation'

$$g(m(x)) = g(x) + 1.$$

This allows a closed formula for iterates of m and, at the same time, a definition of "fractional iterates" of m:

$$m_t(x) := g^{-1}(g(x) + t) \text{ for } t \geq 0.$$

In our cases, g is differentiable and if $p(x) := 1/g'(x)$, then

$$\frac{\partial}{\partial x}(x * y) = \frac{p(x * y)}{p(x)}.$$

secant addition is characterized by $p(x)$ being a polynomial of degree at most 2, (e.g. $p(x) = 1 - x^2$ in the case with $F(x) = \tanh x$) whereas sine-type addition is characterized by another class of functions (e.g., $p(x) = \sqrt{1 - x^2}$ in the case with $F(x) = \sin x$). Other choices of $p(x)$ give rise to algebraic addition:

$$\int_a^x \frac{dt}{p(t)} + \int_a^y \frac{dt}{p(t)} = \int_a^{x*y} \frac{dt}{p(t)}.$$

An interesting example is when $p(x) := \sqrt{(1 - x^2)(1 - k^2x^2)}$ for a fixed $k \in (0, 1)$ (the cases when $k = 0$ and $k = 1$ are covered by the two examples above). The addition defined by this is

$$x * y = \frac{xp(y) + yp(x)}{1 - k^2x^2y^2}$$

and has been well studied; for example, the Jacobi elliptic function $\mathrm{sn}(x, k)$ is a homomorphism ([WW]).

REFERENCES

[A1] J. Aczel, "Lectures on Functional Equations and their Applications", Academic Press, New York, 1966.

[A2] J. Aczel, *The state of the second part of Hilbert's fifth problem*, Bull. Amer. Math. Soc. (N.S.) 20 (1989), no. 2, 153-163.

[A3] J. Aczel, *A Mean Value Property of the Derivative of Quadratic Polynomials – without Mean Values and Derivatives*, Mathematics Magazine, Vol 58, no. 1 (1985) 42-45.

[Au] G. Aumann, Problem #21 for Ninth International Symposium on Functional Equations (1971), *http://riesz.math.klte.hu/ isfe//isfe1971_9/info.php*

[Be] W. Benz *Ein Beltrag zu einem Problem von Hernn Fenyo* Abh. Math. Sem. Univ. Hamburg 57 (1987) 21-25.

[Bo] R. Bojanic, *Une propriete caracteristique des courbes du second degre* Bull Soc. Math. Phys. Serbie, 1, (1949). no. 3-4, 105-111.

[Br] J. Brzdek, *On the Baxter functional equation*, Aequationes Math. 52 (1996), no. 1-2, 105-111.

[BB] J.M. Borwein and P.B. Borwein, "Pi and the AGM", Wiley, New York, 1987.

[BD] N. Brillouet and J. Dhombres *Equations fonctionnelles et recherche de sous-groupes*, Aequationes Math. 31 (1986), no. 2-3, 253-293.

[CMc] R. Coleman and F. McGuinness, *Rational Formal Group Laws*, Pacific J. Math. 147 (1991), no. 1, 25-27.

[K] V.S. Kalnitsky, *Means Generating the Cone Sections and the Third Degree Polynomials*, SPb. Math. Society Preprint 2004-04.

[N1] S. Northshield, *On iterates of Möbius transformations on fields*, Math. Comp. 70 #235 (2000) 1305-1310.

[N2] S. Northshield, *Associativity of the Secant Method*, American Math. Monthly 109 (2002), no. 3, 246-257.

[S1] M. Sablik, *Final part of the answer to Hilbert's question* Functional equations – results and advances, 231-242, Adv. Math. (Dordr.), 3, Kluwer Acad. Publ., Dordrecht, 2002.

[S2] M. Sablik, *On some local topological semigroups*, Aequationes Math. 44 (1992), no. 2-3, 194-219.

[ST] J.H. Silverman and J. Tate, "Rational Points on Elliptic Curves", Springer, New York, 1992.

[VW] P. Volkmann and H. Weigel, *Uber ein Problem von Fenyo*, Aequationes Math. 27 (1984), no. 1-2, 135-149.

[WW] E.T. Whittaker and G.N. Watson, "A Course of Modern Analysis", Fourth Edition, Cambridge Univ. Press, 1952.

E-mail address : samuel.northshield@plattsburgh.edu