A New Parameterization for Ford Circles

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ABSTRACT

Lester Ford introduced Ford circles in 1938 in order to geometrically understand the approximation of an irrational number by rational numbers. We shall construct Ford circles by a recursive geometric procedure. The Ford circles also turn out to be parameterized by the rational numbers. We introduce a new parameterization of the set of Ford circles in terms of triples of relatively prime integers that satisfy a certain equation. This is interesting because we have developed a better approximation for irrational numbers between 0 and 1 and because our new parameterization generalizes to a higher dimension.

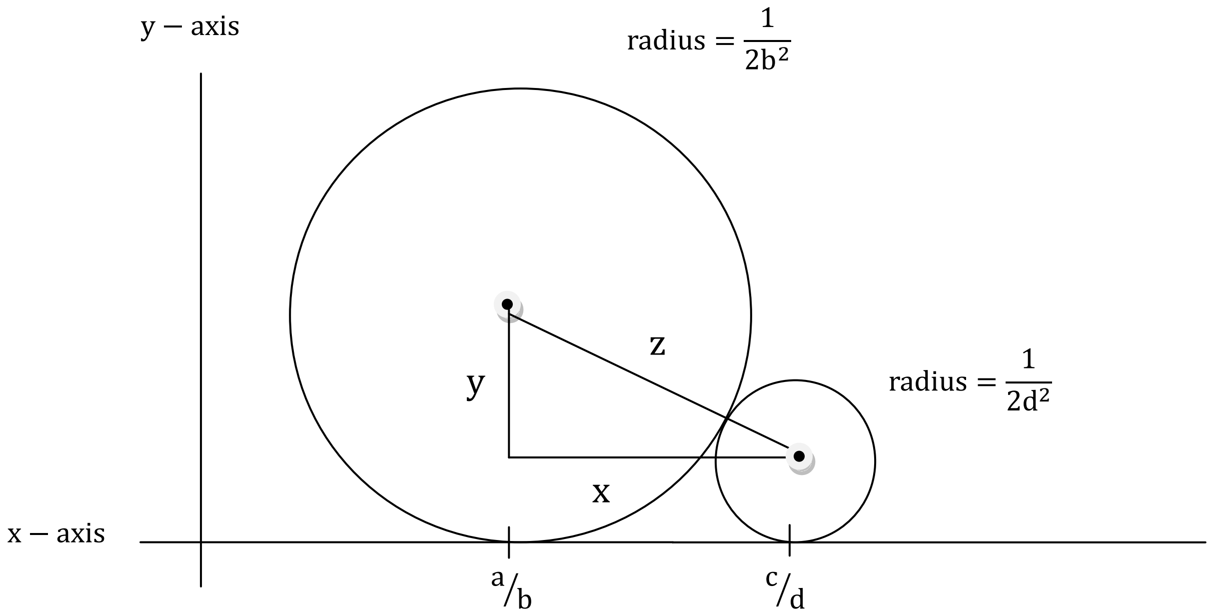
Keywords: Ford circle, Diophantine approximation

INTRODUCTION

Lester Ford introduced Ford circles in 1938 in order to understand the approximation of an irrational number by rational numbers from a geometrical perspective. We shall construct the Ford circles by a recursive geometric procedure. The Ford circles also turn out to be parameterized by the rational numbers (i.e., for any rational number, a/b, in lowest terms, we can construct a circle and radius, $^1/_{2b^2}$, determined by the numbers a and b). We introduce a new parameterization of the set of Ford circles in terms of triples of relatively prime integers x, y, and z that satisfy $x^2 + y^2 + z^2 = (x + y + z)^2$. This is of interest since it seems to generalize to higher dimensions.

GETTING TO KNOW FORD CIRCLES

A Ford circle, denoted $F_{a/b}$, is tangent above the x-axis at a point (a/b,0) with radius $\frac{1}{2b^2}$ where a and b are in **Z** and relatively prime. We will next take a look at two Ford circles that are tangent to one another. For this paper, we will restrict our attention to $0 \leq {}^a/_b \leq 1$. Now let $F_{a/b}$ denote the Ford circle tangent at a point (a/b,0) and $F_{c/d}$ denote the Ford circle tangent at a point (c/d,0). We are working in the x,y plane. The picture below will illustrate.



In order to proceed, we must see that two Ford circles $F_{a/b}$ and $F_{c/d}$ are tangent to one another if and only if $|ad - bc| = 1$. As we can see from the picture above, when two Ford circles are tangent to one another, a right triangle can be drawn as follows: a vertical line, *l*, that is perpendicular to the x-axis can be dropped down from the center of $F_{a/b}$ and a horizontal line, *m*, which is parallel with the x-axis and perpendicular to *l* can be drawn from the center of $F_{c/d}$ to intersect with *l*. Finally, the hypotenuse of our right triangle is drawn by connecting the radii of the circles with a line. Therefore, if we label the sides of the triangle as in the picture above, we can write x, y, and z in terms of a, b, c, and d as follows:

$$x = \frac{c}{d} - \frac{a}{b} \qquad y = \left|\frac{1}{2b^2} - \frac{1}{2d^2}\right| \qquad z = \frac{1}{2b^2} + \frac{1}{2d^2}$$

Since we have a right triangle, we know $x^2 + y^2 = z^2$. We will use these facts to prove our first theorem. This has already been proven by others, but I thought it was worthwhile to show.

Theorem 1: Two Ford circles, $F_{a/b}$ and $F_{c/d}$, are tangent to one another if and only if $|ad - bc| = 1$.

Proof: Assume $F_{a/b}$ and $F_{c/d}$ are tangent to one another. So, by the Pythagorean Theorem, $x^2 + y^2 = z^2$.
When we substitute the values of x, y and z in terms of a, b, c, and d, the following happens:

$$\left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2$$

$$\Longleftrightarrow \quad \frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} + \frac{1}{4b^4} - \frac{1}{2b^2d^2} + \frac{1}{4d^4} = \frac{1}{4b^4} + \frac{1}{2b^2d^2} + \frac{1}{4d^4}$$

$$\Longleftrightarrow \quad \frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} + \cancel{\frac{1}{4b^4}} - \frac{1}{2b^2d^2} + \cancel{\frac{1}{4d^4}} = \cancel{\frac{1}{4b^4}} + \frac{1}{2b^2d^2} + \cancel{\frac{1}{4d^4}}$$

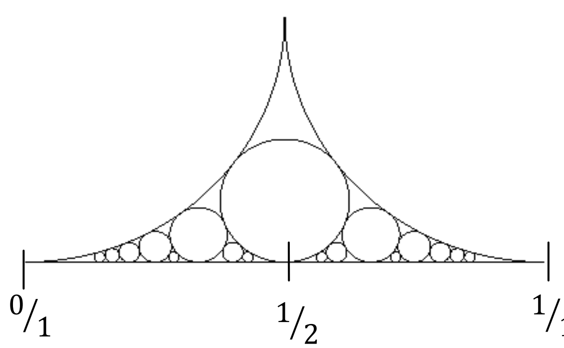
$$\Longleftrightarrow \quad \frac{a^2}{b^2} + \frac{c^2}{d^2} - \frac{2ac}{bd} - \frac{1}{b^2d^2} = 0 \quad \Longleftrightarrow \quad \frac{a^2d^2}{b^2d^2} + \frac{b^2c^2}{b^2d^2} - \frac{2abcd}{b^2d^2} - \frac{1}{b^2d^2} = 0$$

$$\Longleftrightarrow \quad a^2d^2 + b^2c^2 - 2abcd - 1 = 0 \quad \Longleftrightarrow \quad a^2d^2 - 2abcd + b^2c^2 = 1$$

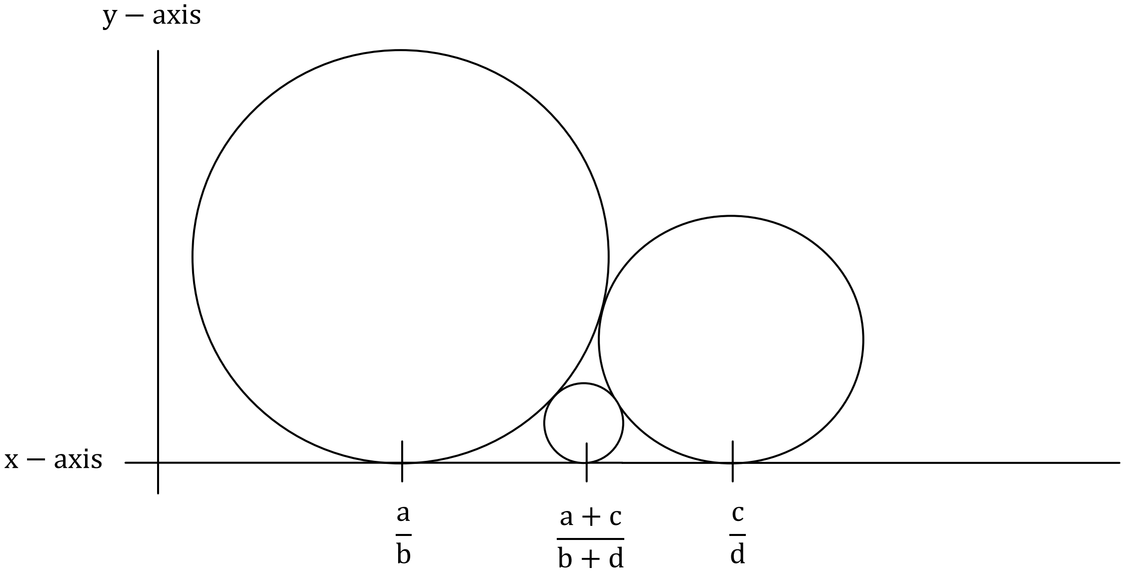
$$\Longleftrightarrow \quad (ad - bc)(ad - bc) = 1 \quad \Longleftrightarrow \quad (ad - bc)^2 = 1 \quad \Longleftrightarrow \quad |ad - bc| = 1.$$

We have successfully shown that two Ford circles, $F_{a/b}$ and $F_{c/d}$, are tangent to one another if and only if $|ad - bc| = 1$. ∎

RECURSIVE CONSTRUCTION FOR FORD CIRCLES



We start with $F_{1/1}$ and $F_{0/1}$ which are clearly tangent to one another and both have radius 1/2. Consider the following:



Continuing the recursive construction, we assume two Ford circles, $F_{a/b}$ and $F_{c/d}$, are tangent to one another. We then draw another Ford circle, $F_{a+c/b+d}$, that is tangent to both of the original two. We can then draw yet another smaller Ford circle that is tangent to both $F_{a+c/b+d}$ and $F_{a/b}$ and another that is tangent to $F_{c/d}$ and $F_{a+c/b+d}$. We can continue this process over and over again. We can draw more Ford circles that are tangent to two bigger ones infinitely many times. The beginning of this process is illustrated in the picture above. To make this more rigorous, we will prove another theorem. Now, let $F_{a/b} \; || \; F_{c/d}$ denote the two Ford circles sharing a tangent line.

Theorem 2: $F_{a/b} \, || \, F_{c/d} \Rightarrow F_{a+c/b+d} \, || \, F_{a/b}$ and $F_{a+c/b+d} \, || \, F_{c/d}$.

Proof: Assume $F_{a/b} \; | \; | \; F_{c/d}$. Then $|ad - bc| = 1$.

So, $|ad - bc| = |ab + ad - ab - bc| = |a(b + d) - b(a + c)|$.

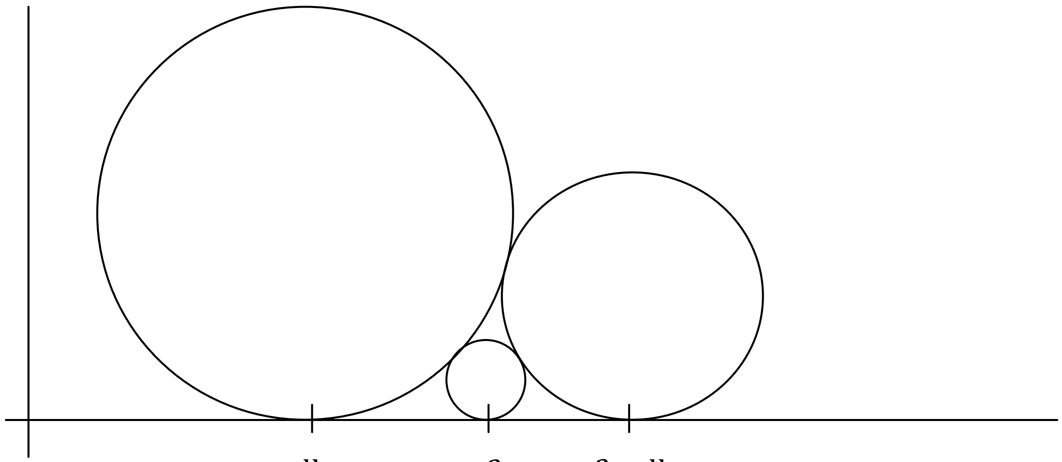
Therefore, $|a(b + d) - b(a + c)| = 1$, so $F_{a/b} \; || \; F_{b+d/a+c}$.

Similarly, $|ad - bc| = |ad + cd - bc - cd| = |(a + c)d - (b + d)c|$.

Therefore, $|(a + c)d - (b + d)c| = 1$ so, $F_{a+c/b+d} \; || \; F_{c/d}$.

$\blacksquare$

We have successfully shown that if two Ford circles $F_{a/b}$ and $F_{c/d}$ share a tangent line, then the Ford circle, $F_{a+c/b+d}$, is tangent to both $F_{a/b}$ and $F_{c/d}$. This also shows that every circle that arises from the recursive construction is a Ford circle. In other words, when we start with two Ford circles and draw another circle between them that is tangent to both of them, the smaller circle will always be a Ford circle. Next, we will show that every Ford circle arises from the recursive method. Consider the following picture:



Theorem 3: Every Ford circle arises from the recursive construction.

Proof: We will show by induction.
Let $\mathbf{S} = \{ {}^{a}/_{b} : F_{a/b} \text{ arises from recursive method} \}$.
Let p(n) be the statement," ${}^{j}/_{k} \in S$ for every ${}^{j}/_{k} \in [0,1], GCD(j,k) = 1, k < n$."
We know that p(2) is true since ${}^{0}/_{1}, {}^{1}/_{1} \in \mathbf{S}$.
Now suppose p(n) is true.
Let a/n, a rational in lowest terms, represent a Ford circle, $F_{a/n}$.
(We must show that a/n is in **S** to show p(n) is true for $k \le n$ which implies p(n+1) is true).
In order to show a/n is in **S**, we must show that there exist two larger Ford circles in which $F_{a/n}$ is tangent to both of them. We will show p(n) is true for all n greater than or equal to 2.
So, we know that $GCD(a,n) = 1$.
[3] According to Saracino et al. (1980) , $\exists\, x, y \in \mathbf{N}: ax + ny = gcd(a,n) = 1$.
Since there is a unique $k \in \mathbf{Z}$ such that $0 \le kn + x \le n$, then letting u=ka-y and v=kn+x, av-nu=1, u, and v are natural numbers, with $v \le n$.

This means we have another Ford circle, $F_{u/v}$, such that $u \le v \le n$ that is tangent to $F_{a/n}$.

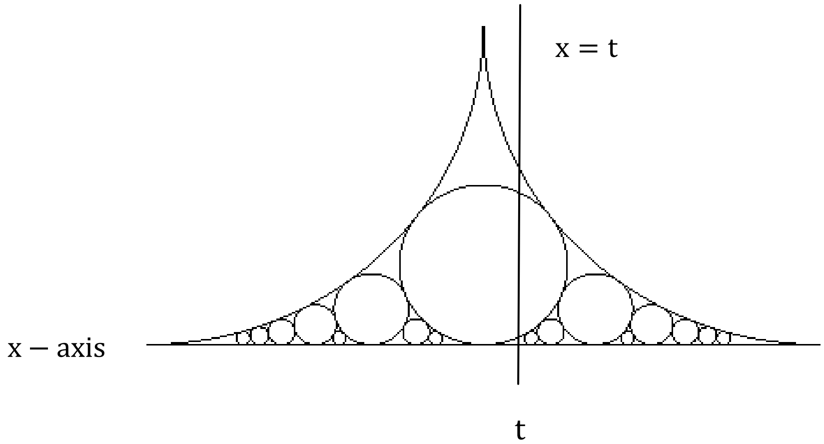
Notice $F_{a-u/n-v} \,||\, F_{a/n}$ since $|(a-u)(n) - [(n-v)(a)]| = |av - nu| = 1$
and $F_{a-u/n-v} \,||\, F_{u/v}$ since $|(a-u)(v) - [(n-v)(u)]| = |av - nu| = 1$.

Therefore, all three of the circles are tangent to one another. More importantly, we have shown that the Ford circle represented by a/n is tangent to both of the circles represented by u/v and (a-u)/(n-v). This shows that a/n is in **S.** So, we have successfully shown that p(n) is true for all n greater than or equal to 2.

We have shown that every Ford circle arises from the recursive construction. We will now take a look at Diophantine approximation and the connections between this and Ford circles.

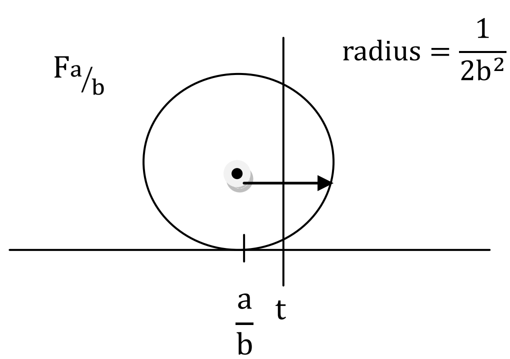
DIOPHANTINE APPROXIMATION

Diophantine approximation is done when an irrational number is approximated by a rational one. For example, many people approximate π using the fraction 22/7. Ford circles allow us to see Diophantine approximation from a geometric perspective. We will also take a look at this picture:



If we let t be an irrational number, we know that x=t will never bisect a Ford circle because every Ford circle is tangent to the x-axis at a point (s, 0), where s is a rational number. Therefore, we know that a line x=t will pass through infinitely many Ford circles. This is the picture that Ford circles provide the viewer when learning about Diophantine approximation. I will next refer to a book written by Ivan Niven entitled, *Diophantine Approximations*. **[1]** Niven et al. (1963) state, "given any irrational number θ, there are infinitely many rational numbers a/b, where a, b >0, such that $\left|\theta - {}^{a}/_{b}\right| < {}^{1}/_{b^2}$ is true. In other words, given an irrational number $\theta > 0$ we can pick a rational number, ${}^{a}/_{b}$ with $b \neq 0$ so that the inequality is true.

If we take a look at another picture of a Ford circle, we can see a connection.



Note that a, b are relatively prime integers and t is an irrational number.

We can see from our picture that the following inequality is always true. This is interesting because, as we can see from the picture, we have developed a better approximation for irrationals between 0 and 1 than the Diophantine approximation in Niven's book.

Our approximation:

$$\left| t - \frac{a}{b} \right| < \frac{1}{2b^2}$$

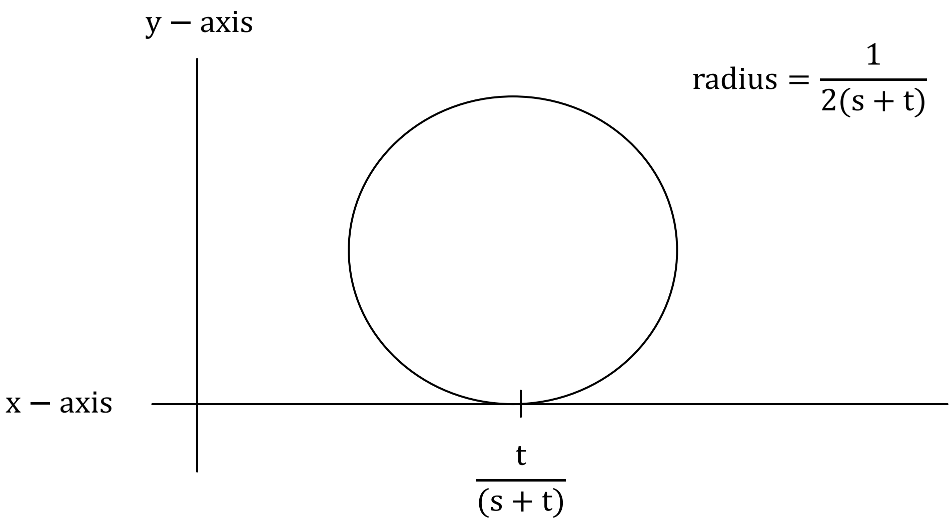
Niven's approximation:

$$\left| t - \frac{a}{b} \right| < \frac{1}{b^2}$$

We will now consider another class of circles and show that under some conditions these are Ford circles.

NEW PARAMETERIZATION

Let $< s, t >, s, t \in \mathbf{Z}$, denote a circle.



If $< s, t > is\ a\ Ford\ circle, then\ \exists\ a, b \in \mathbf{R}$: $\frac{a}{b} = \frac{t}{(s+t)}$ and $\frac{1}{2b^2} = \frac{1}{2(s+t)}$.

We must see exactly when a circle <s,t> is a Ford circle. We will do this using the two above equations. This will allow us to make further connections later in this paper. This is new research that I have been working on with the help of one of my professors.

$$\frac{1}{2b^2} = \frac{1}{2(s+t)} \Rightarrow \frac{1}{b^2} = \frac{1}{(s+t)} \Rightarrow b = \sqrt{(s+t)}$$

$$\frac{a}{\sqrt{(s+t)}} = \frac{t}{(s+t)} \Rightarrow a = \frac{t(s+t)^{1/2}}{(s+t)} \Rightarrow a = \frac{t}{(s+t)^{1/2}}$$

$$\frac{a}{b} = \frac{t}{(s+t)} \Rightarrow t = \frac{a}{b}(s+t) = \frac{a}{b}(b^2) = ab, \quad \text{so } t = ab$$

$$t = ab \Longrightarrow \frac{a}{b} = \frac{ab}{s + ab} \Longrightarrow ab = \frac{a}{b}(s + ab) = \frac{a}{b}(s) + a^2 \Longrightarrow \frac{a}{b}(s) = ab - a^2$$

$$\Longrightarrow s = \frac{b}{a}(ab - a^2) = b^2 - ab.$$ So, $< s, t >$ *is a Ford circle when* $s = b^2 - ab$ and $t = ab$.

Next we will examine some sets to make further connections.

CURIOUS CONNECTIONS

Let $\mathbf{A} = \{(a, b, c) \in \mathbf{Z}^3 : GCD(a, b, c) = 1, (a + b + c)^2 = a^2 + b^2 + c^2 \text{ and } abc < 0\}$.

Note that the condition $abc < 0$ and the fact $(a + b + c)^2 = a^2 + b^2 + c^2$ ensure exactly one of a, b, and c is negative.
Some triples that are in the set **A** are:
(6, -2, 3), (20, -4, 5), (36, -20, 45), (21, 28, -12), (12, -3, 4), (56, -7, 8), and (14, -10, 35)
It will be useful later to look at certain numerical examples to appreciate how interesting these connections are.

Let $\mathbf{U} = \{(xy, x^2 - xy, y^2 - xy) : x, y \in \mathbf{Z}, GCD(x, y) = 1\}$.

Theorem 4: The sets **A** and **U** are equal.

Proof: First, we will show that **A** is a subset of **U**.
Let $z \in \mathbf{A}$, then $z = (a, b, c)$ for some $a, b, c \in \mathbf{Z}$, $GCD(a, b, c) = 1, (a + b + c)^2$.
So, we know $(a + b + c)^2 = a^2 + b^2 + c^2 \Longrightarrow ab + ac + bc = 0 \Longrightarrow c = \dfrac{-ab}{a + b}$
$\Longrightarrow (ka, kb, kc)$ is proportional to (a, b, c) where $k = (a + b)$.
i.e., (a, b, c) is proportional to $(a(a + b), b(a + b), -ab)$.
Suppose $g = GCD(a, b)$ and define $m = a/g$ and $n = b/g$, then m and n are relatively prime and $(m(m + n), n(m + n), -mn)$ is proportional to (a, b, c).
They both have relatively prime integers, so the only thing we need to check to show they are equal is to see if (m(m+n),n(m+n),-mn) has two positive components and one negative component. We can do this by multiplying the three components of (m(m+n),n(m+n),-mn) together to see if we get a negative product. After doing the algebra it is clear to see that (m(m+n))(n(m+n))(-mn) = $-m^2n^2(m + n)^2$, so it is always true that two components are positive and one is negative. Therefore, since (a, b, c) = (m(m+n),n(m+n),-mn), z is an element of **U** and **A** is a subset of **U.** (We can let x=m+n and y=m from our set **U** to see that z is in **U**).

We will next prove that **U** is a subset of **A**.
Let $z \in \mathbf{U}$, then $z = (xy, x^2 - xy, y^2 - xy)$ for some $x, y \in \mathbf{Z}$.
Seeing if $z \in \mathbf{A}$: $[(xy) + (x^2 - xy) + (y^2 - xy)]^2 = (x^2 - xy + y^2)^2 =$

$$x^4 - x^3y + x^2y^2 - x^3y + x^2y^2 - xy^3 + x^2y^2 - xy^3 + y^4 =$$

$$(x^2y^2) + (x^4 - 2x^3y + x^2y^2) + (y^4 - 2xy^3 + x^2y^2) = (xy)^2 + (x^2 - xy)^2 + (y^2 - xy)^2$$

Let $z = \left((xy), (x^2 - xy), (y^2 - xy)\right) = (q, r, s)$, then $(q + r + s)^2 = q^2 + r^2 + s^2$.

Therefore, $z \in \mathbf{A}$ and $\mathbf{U} \subseteq \mathbf{A}$, so $\mathbf{A} = \mathbf{U}$.

Take a look at a numerical example from earlier: $(6, -2, 3) = \left((2)3, 2^2 - 3(2), 3^2 - (2)3\right)$
Therefore, (6, -2, 3) is also in **U**. Our next Theorem will highlight an interesting fact about **A** and **U**.

Now let $\mathbf{T} = \{(d_1, d_2, d_3) \in \mathbf{Z}^3 : \mathrm{GCD}(d_1, d_2, d_3) = 1, d_i + d_j = \text{perfect square if } i \neq j\}$.

Theorem 5: The set **U** is a subset of **T**.

Proof: Let $u \in \mathbf{U}$, then $u = (ab, a^2 - ab, b^2 - ab)$ for some $a, b \in Z$, $\mathrm{GCD}(a, b) = 1$.

We must check to see if u is an element of **T** and we will do so by checking each combination of two of three components by adding them together to see if the sum is a perfect square.

$(ab) + (a^2 - ab) = a^2$ √ $\qquad (ab) + (b^2 - ab) = b^2$ √

$$(a^2 - ab) + (b^2 - ab) = a^2 - 2ab + b^2 = (a - b)^2 \quad \surd$$

Therefore $u \in \mathbf{T}$ and $\mathbf{U} \subseteq \mathbf{T}$.

Looking at a numerical example from our original set **A**, we can easily see that it is also in **T**:
e.g., $(6, -2, 3)$: $6 - 2 = 4 = 2^2$, $6 + 3 = 9 = 3^2$, and $3 - 2 = 1 = 1^2$

Because we know that **A**=**U** and **U** is a subset of **T**, we can conclude that **A** is also a subset of **T**.

Our next theorem will illustrate our new parameterization of Ford circles.

Theorem 6: The set of Ford circles =
$\{< s, t >: s, t, u \ rel. prime, s^2 + t^2 + u^2 = (s + t + u)^2, stu < 0\}$.

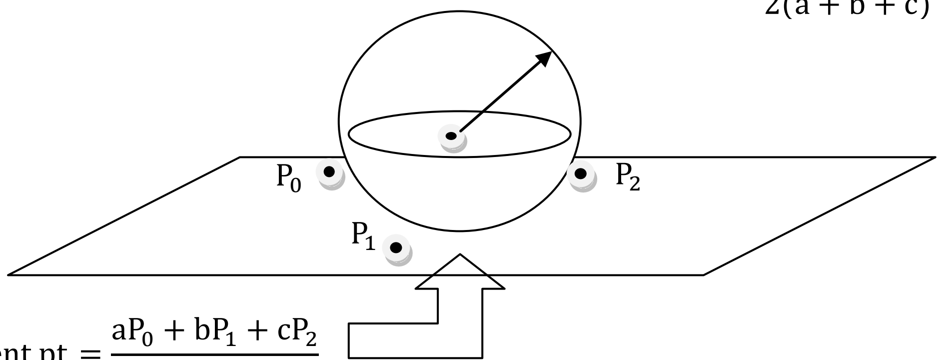
Proof: We know the set of Ford Circles $= \{< xy, x^2 - xy >: x, y \ rel. prime\}$
$= \{< s, t >: (s, t, u) = (xy, x^2 - xy, y^2 - xy), x, y \text{ rel. prime}\}$
$= \{< s, t >: (s, t, u) \in \mathbf{U}\} = \{< s, t >: (s, t, u) \in \mathbf{A}\}$
$= \{< s, t >: s, t, u \ rel. prime, s^2 + t^2 + u^2 = (s + t + u)^2, stu < 0\}$.

This helps because our new parameterization for Ford circles can be generalized to a family of spheres in the third dimension. **[2]** My professor, Dr. Sam Northshield, has proved this fact. If we express our set of Ford circles in this different way, we can generalize from 2D to 3D. Our next picture will illustrate this concept.

3rd DIMENSION

$< a, b, c >$ *denotes this sphere*

$$\text{radius} = \frac{1}{2(\mathrm{a}+\mathrm{b}+\mathrm{c})}$$



$$\text{tangent pt.} = \frac{\mathrm{aP_0} + \mathrm{bP_1} + \mathrm{cP_2}}{\mathrm{a}+\mathrm{b}+\mathrm{c}}$$

(P_0, P_1, P_2 are vertices of equilateral triangle in plane, side $= 1$)

$\{< a, b, c >: a, b, c, d \ rel. prime \ (\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d})^2 = \mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2+\mathrm{d}^2\}$

This set of spheres shares many properties with the set of Ford circles.

CONCLUSION

It is interesting to see the connections between Ford circles and Diophantine approximation. I believe it is interesting because we have developed a better approximation for irrational numbers between 0 and 1 and our new parameterization generalizes to a higher dimension. I plan to continue my research and build on this project to see what I might discover.

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