2010

Sums across Pascal's triangle modulo 2

Sam Northshield
SUNY Plattsburgh

Recommended Citation
Northshield, S. (2010). Sums across Pascal's triangle modulo 2. *Congressus Numerantium, 200*, 35-52.

PREPRINT
January 23, 2007

SUMS ACROSS PASCAL'S TRIANGLE MODULO 2

S. Northshield

Department of Mathematics, SUNY-Plattsburgh
Plattsburgh, NY 12901

Abstract. We consider sums of the binomial coefficients $C(i+j,i)$ modulo 2 over lines $ai+bj=n$. Many interesting sequences (old and new) arise this way.

1. Introduction.

Two well known facts about summing along rows and other lines across Pascal's triangle are

$$\sum_{i+j=n} \binom{i+j}{i} = 2^n,$$

and

$$\sum_{2i+j=n} \binom{i+j}{i} = F_{n+1} \sim C\phi^n$$

(here, of course, F_n is the n-th Fibonacci number and ϕ is the "Golden ratio" 1.618...).

In a number of papers, generalizations have been considered. For example, Raab [R] was apparently the first to investigate sums

$$c_n := \sum_{ai+bj=n} \binom{i+j}{i}$$

for integral and relatively prime a and b; see also the paper by Green [Gre]. To understand the asymptotic behavior of the sequence (c_n), note that it satisfies the recurrence

$$c_{n+a+b} = c_{n+a} + c_{n+b}.$$

Thus (c_n) has a Binet type formula

$$c_n = \sum_i A_i \gamma_i^n$$

where $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ are solutions of the characteristic equation $x^{a+b} = x^a + x^b$. This equation has a unique solution γ of largest absolute value and, furthermore, that solution is positive and real. By the Binet type formula,

$$c_n \sim A\gamma^n$$

for some constant A.

AMS 2000 Subject Classification: Primary 11B65; Secondary 11B39, 11B50, 11C20.
Key words and phrases: Binomial coefficients, Fibonacci numbers, matrices.

The fact that γ is unique, positive, and real follows from the Perron-Frobenius theorem – a positive 'aperiodic' matrix has a unique eigenvalue of largest absolute value which is positive and real. It is possible to construct directed graphs whose (positive and 'aperiodic') adjacency matrices have characteristic equations $x^{a+b} = x^a + x^b$.

If we consider the problem of summing over Pascal's triangle modulo 2, then we may not resort to calculus. To fix our notation, let $\binom{n}{k}_2$ be the integer 0 or 1 according (resp.) as $\binom{n}{k} \equiv 0$ or 1 (mod 2). That is,

$$\binom{n}{k}_2 := \begin{cases} 0 & \text{if } 2 \mid \binom{n}{k}, \\ 1 & \text{if } 2 \nmid \binom{n}{k}. \end{cases}$$

It turns out that sums of the form

$$\sum_{ai+bj=n} \binom{i+j}{i}_2$$

have been considered when $a = b = 1$ and when $a = 2, b = 1$. We shall refer to these as the $(1,1)$ and $(2,1)$ cases respectively.

We first shall review what is known of these two cases. Although the results here are known, the proofs are perhaps new. Then we shall consider analogous results in the $(3,1)$ case. I believe that the results here are new. Finally we shall consider the general (a,b) case and directions for further research.

I wish to thank an anonymous referee for many useful suggestions.

2. The $(1,1)$ and $(2,1)$ cases.

We consider first Gould's sequence [Sl:A007318]:

$$g_n := \sum_{i+j=n} \binom{i+j}{i}_2 = \sum_i \binom{n}{i}_2.$$

The first few terms of the sequence are

1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, 2, 4, 4, 8, 4, 8, 8, 16, 4, 8, 8, 16, 8, 16, 16, 32, 2, ...

This sequence has been referred to as "self-similar" [Sc]; indeed, it is a fixed point of the morphism $1 \mapsto 12$, $2 \mapsto 24$, $4 \mapsto 48, \ldots$. It is easy to see that g_n is always a power of 2 (and so $\log_2 g_n$ is an integer sequence). Modulo two, the sequence $\log_2 g_n$ is the Morse-Thue sequence [Sc], invariant under $0 \mapsto 01$ and $1 \mapsto 10$:

0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, ...

The first few terms of $\log_2 g_n$ are

0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5, 1, ...

This agrees with a well known sequence [Sl: A000120]; $x_n :=$ the number of ones in the binary representation of n. We shall give a proof of this based on the following lemma due to Kummer in 1852; see [GKP, exercise 5.36].

Lemma 2.1. *Let p^k be the largest power of the prime p that divides $\binom{i+j}{i}$. Then k is the number of carries that occur when i is added to j in the radix p number system. In particular, $\binom{i+j}{i}_2 = 1$ if and only if i and j have no ones in the same place in their binary representations.*

The next theorem appears in [Gra] and has been attributed there to a 1899 paper by Glaisher.

Theorem 2.2. $g_n = 2^{x_n}$

Proof. By Lemma 2.1, $\binom{n}{i}_2 = 1$ if and only if i and $n-i$ have no ones in the same place in base-2 notation; i.e., the ones in the binary expansions of i and $n-i$ partition the ones in the binary representation of n. This occurs if and only if the ones in the base-2 notation of i occur in some subset of the locations of the ones in the base-2 notation of n. There are 2^{x_n} such subsets. □

It is very clear that x_n, the number of ones in the base-2 representation of n, satisfies the recurrence relations

$$x_{2n} = x_n$$
$$x_{2n+1} = x_n + 1.$$

By Theorem 2.2, recurrence relations for (g_n) follow.

Corollary 2.3. *The sequence (g_n) may be defined recursively by: $g_0 = 1$, $g_1 = 2$, and*

$$g_{2n} = g_n$$
$$g_{2n+1} = 2g_n.$$

Let $G(x) := \sum g_n x^n$ be the generating function for Gould's sequence. $G(x)$ then satisfies a functional equation and has a product formula.

Corollary 2.4.

$$G(x) = (1+2x)G(x^2) = \prod_n (1+2x^{2^n}).$$

Proof. Using Corollary 2.3,

$$\begin{aligned}(1+2x)G(x^2) &= \sum_n g_n x^{2n} + \sum_n 2g_n x^{2n+1} \\ &= \sum_n g_{2n} x^{2n} + \sum_n g_{2n+1} x^{2n+1} = G(x).\end{aligned}$$

Replacing x by x^{2^n},

$$G(x^{2^n}) = (1+2x^{2^n})G(x^{2^{n+1}})$$

and it follows that

$$G(x) = (1+2x)G(x^2) = (1+2x)(1+2x^2)G(x^4) = \dots$$

and the product formula follows immediately. □

Finally, one can arrange Gould's sequence, starting with g_1, in an array:

$$\begin{array}{cccccccccccccccc}
2 \\
2 & 4 \\
2 & 4 & 4 & 8 \\
2 & 4 & 4 & 8 & 4 & 8 & 8 & 16 \\
2 & 4 & 4 & 8 & 4 & 8 & 8 & 16 & 4 & 8 & 8 & 16 & 8 & 16 & 16 & 32 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

Evidently, in this array, the numbers are constant along each column; this follows easily from the following corollary.

Corollary 2.5 *If* $2^n > j$*, then* $g(2^n + j) = 2g(j)$*. Hence, if* $2^n, 2^m > j$*, then* $g(2^n + j) = g(2^m + j)$*.*

Proof. Suppose $2^n > j$. Then $x(2^n+j) = x(j)+1$ and, by Theorem 2.2, $g(2^n+j) = 2g(j)$. □

We remark (and leave it as an exercise for the reader) that the row sums for the array are of the form $2 \cdot 3^{n-1}$.

Now, we consider

$$b_n := \sum_{2i+j=n} \binom{i+j}{i}_2 = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}_2.$$

The first few terms of b_n are:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 7, 4, 5, 1, 6, 5, 9, 4, 11, ...$$

The sequence formed by putting a zero at the beginning:

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 7, 4, 5, 1, 6, 5, 9, 4, ...$$

is called 'Stern's diatomic sequence' and possesses an amazing array of properties; see [Sl:A002487] for a concise summary and many references. Our sequence (b_n) of course shares many of these properties some of which we will elucidate below.

As for Gould's sequence (g_n), the sequence (b_n) has a combinatorial interpretation. A "hyperbinary" representation of n is a vector $(e_k, e_{k-1}, ..., e_1, e_0)$ with entries $e_i \in \{0, 1, 2\}$ (the number k is not fixed) such that $\sum e_i 2^i = n$. We let $H(n)$ denote the set of hyperbinary representations of n. For example, $(2, 1, 0, 2, 1, 1) \in H(91)$. Further, we may write

$$(2, 1, 0, 2, 1, 1) = 2 \cdot (1, 0, 0, 1, 0, 0) + (1, 0, 0, 1, 1)$$

where, of course, (1,0,0,1,0,0) and (1,0,0,1,1) are ordinary binary representations of 36 and 19 respectively. Thus

$$2 \cdot 36 + 19 = 91$$

and, furthermore, since the locations of 2's and 1's do not coincide,

$$\binom{36+19}{36}_2 = 1$$

by Lemma 2.1. In general, it is clear that there is a one-to-one correspondence between elements of $H(n)$ and pairs of numbers (i, j) such that $2i + j = n$ and $\binom{i+j}{i}_2 = 1$. Hence:

Theorem 2.6. $b_n = |H(n)|$ *where* $H(n)$ *is the set of vectors* $(e_k, e_{k-1}, ..., e_1, e_0)$ *with entries* $e_i \in \{0, 1, 2\}$ *(the number* k *is not fixed) such that* $\sum e_i 2^i = n$.

We may now derive recurrence relations for b_n.

Corollary 2.7 *The sequence* (b_n) *may be defined recursively by:* $b_0 = b_1 = 1$, *and*

$$\begin{aligned} b_{2n+1} &= b_n \\ b_{2n} &= b_n + b_{n-1}. \end{aligned}$$

Proof. Let $(e_k, e_{k-1}, ..., e_1, e_0) \in H(2n + 1)$. Then $e_0 = 1$ and $(e_k, e_{k-1}, ..., e_1) \in H(n)$. This defines a bijection between $H(2n + 1)$ and $H(n)$ and so, by Theorem 2.6, $b_{2n+1} = b_n$. If, on the other hand, $(e_k, e_{k-1}, ..., e_1, e_0) \in H(2n)$, then $e_0 = 0$ or 2. If $e_0 = 0$ then $(e_k, e_{k-1}, ..., e_1) \in H(n)$ and so the elements of $H(2n)$ with $e_0 = 0$ are in one-to-one correspondence with the elements of $H(n)$. Similarly, if $e_0 = 2$ then $(e_k, e_{k-1}, ..., e_1) \in H(n - 1)$ and so the elements of $H(2n)$ with $e_0 = 2$ are in one-to-one correspondence with the elements of $H(n - 1)$. By Theorem 2.6, $b_{2n} = b_n + b_{n-1}$. □

Let $B(x) := \sum b_n x^n$ be the generating function for (b_n). $B(x)$ then satisfies a functional equation and has a product formula.

Corollary 2.8

$$B(x) = (1 + x + x^2)B(x^2) = \prod_n (1 + x^{2^n} + x^{2^{n+1}}).$$

Proof. Using Corollary 2.7,

$$\begin{aligned} (1 + x + x^2)B(x^2) &= \sum_n b_n x^{2n} + \sum_n b_n x^{2n+1} + \sum_n b_n x^{2n+2} \\ &= \sum_n b_n x^{2n} + \sum_n b_{2n+1} x^{2n+1} + \sum_n b_{n-1} x^{2n} \\ &= \sum_n b_{2n} x^{2n} + \sum_n b_{2n+1} x^{2n+1} \\ &= \sum_n b_n x^n = B(x). \end{aligned}$$

Iterating this formula as in the proof of Corollary 2.4, the product formula follows. □

As before, we write the terms of the sequence in an array, starting with b_1:

1																				
2	1																			
3	2	3	1																	
4	3	5	2	5	3	4	1													
5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1					
6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	. . .
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.				

Numerous patterns are apparent. Among them:

$$b(2^n - 1) = 1,$$

and

$$b(2^n) = n + 1.$$

Both of these follow easily from the recurrences for (b_n) [Corollary 2.7].

The rows are palindromic:

$$b(2^n - 1 + j) = b(2^{n+1} - 1 - j) \text{ if } 2^n > j.$$

This follows from Corollary 2.7 and induction. Visually, it is clear from the symmetry of 'Stern's diatomic array' for the 'augmented' sequence. That is, attach a one before each row and arrange the entries as follows:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1.

The recurrence relations for (b_n) show that columns are constant and new entries in a row are sums of entries in the previous rows. See [Sl: A002487] and references therein.

We have three properties of the first array (which we present without proof):

a) The Fibonacci numbers make an appearance:

$$\max\{b_k : 2^n \leq k < 2^{n+1}\} = b(1 + (2^{n+2} - (-1)^{n+2})/3) = F_{n+2}.$$

b) The columns of the array are arithmetic sequences and the differences correspond to the original sequence:

$$b(2^{n+1} + j) = b(2^n + j) + b(j) \text{ if } 2^n > j.$$

c) The row sums are powers of 3.

The sequence (b_n) has some remarkable number theoretic properties. To fix notation, we write $a \perp b$ if a and b are relatively prime. A property noted by Dijkstra [Sl: A002487] is that

$$b(2^n + j) \perp b(2^{n+1} - 1 - j)$$

which, by the palindromic nature of the rows, is contained in the fact that

$$b_n \perp b_{n+1}.$$

The fractions b_n/b_{n-1} are then automatically in lowest term and form a sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{4}, \frac{5}{3}, \frac{2}{5}, \frac{5}{2}, \frac{3}{5}, \frac{4}{3}, \frac{1}{4}, \frac{5}{1}, \ldots$$

which has the property that every positive rational number is represented exactly once. A charming proof of this can be found in [CW] based on the construction of a tree. We shall present another proof which is essentially contained in [GKP, section 4.5]. The key is to represent the recurrence relations for (b_n) in terms of matrices. Let $v_0 := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and, for $n \geq 1$, $v_n := \left(\begin{smallmatrix} b_n \\ b_{n-1} \end{smallmatrix}\right)$. Define matrices

$$A_0 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \text{ and } A_1 := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right).$$

Then the recurrence relations for (b_n) can be written as

$$A_0 v_n = v_{2n}$$
$$A_1 v_n = v_{2n+1}.$$

These matrices are generators of the group corresponding to the "Farey tesselation" of the hyperbolic half-plane which gives a geometric interpretation of continued fractions; see [Se] for details. We shall use the connection with continued fractions in our proof.

Theorem 2.9. *For every n, $b_n \perp b_{n-1}$. The sequence of fractions b_n/b_{n-1} includes every positive rational number exactly once.*

Proof. With A_0 and A_1 defined as above, note that if $n = \sum_{k=0}^{r} e(k)2^k$, $(e(k) \in \{0,1\})$ is the usual binary representation of n then

$$v_n = A_{e(0)} A_{e(1)} \cdots A_{e(r)} v_0.$$

Since the entries of v_0 are relatively prime and since A_0 and A_1 preserve that property, it follows by an induction argument that the entries of v_n are relatively prime (i.e., $b_n \perp b_{n-1}$).

We may define an action of 2x2 matrices on real numbers by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(x) := \frac{ax+b}{cx+d}.$$

It is easy to verify the 'modular property'

$$A(B(x)) = (AB)(x).$$

Let $C := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that $C(x) = \frac{1}{x}$, $C^{-1} = C$, and $CA_0C = A_1$.

Let s be a positive rational number. It can be expressed uniquely as a finite continued fraction

$$s = [a_0; a_1, a_2, ..., a_k] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

where k is odd (since every continued fraction ending with 1 can be shortened to one ending with something else e.g., [5,2,1]=[5,3], [5,1,2]=[5,1,1,1]). But then also

$$s = (A_0^{a_0} C A_0^{a_1} C A_0^{a_2} C \cdots C A_0^{a_k} C)(\infty)$$

which, since k is odd, $C^{-1} = C$ and $CA_0C = A_1$, can be written

$$s = (A_0^{a_0} A_1^{a_1} A_0^{a_2} \cdots A_1^{a_k})(\infty).$$

However, there exists n so that

$$v_n = A_0^{a_0} A_1^{a_1} A_0^{a_2} \cdots A_1^{a_k} v_0$$

and thus $s = b_n/b_{n-1}$. This shows that every s is represented in the sequence of fractions b_n/b_{n-1}. The uniqueness of the continued fraction expansion of s shows that s is represented by only one b_n/b_{n-1}. □

We give a couple examples illustrating the bijection between the positive rationals and the integers.

Start with the positive rational $\frac{4}{11}$. Its continued fraction representation is

$$\frac{4}{11} = \cfrac{1}{2 + \cfrac{1}{1+\frac{1}{3}}} = [0; 2, 1, 3].$$

Since $C(x) = 1/x$ and $A_0(x) = x + 1$,

$$\frac{4}{11} = CA_0^2 C A_0 C A_0^3 C(\infty)$$

which can be rewritten as

$$\frac{4}{11} = A_1^2 A_0 A_1^3(\infty).$$

Because $b_n \perp b_{n-1}$,

$$\begin{pmatrix} 4 \\ 11 \end{pmatrix} = A_1^2 A_0 A_1^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and since (1,1,1,0,1,1) represents 59 in binary,

$$v_{59} = A_1^2 A_0 A_1^3 v_0 = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

Now we give an example of the reverse procedure; starting with a number, say 14, we find v_{14}. In binary, $14 = (1, 1, 1, 0)$ and so

$$v_{14} = A_0 A_1^3 v_0 = A_0 C A_0^3 C v_0.$$

The corresponding number is

$$A_0 C A_0^3 C(\infty) = 1 + \frac{1}{3 + \frac{1}{\infty}} = \frac{4}{3}.$$

Hence

$$v_{14} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Note that the procedure (let's call it ω) that took $\frac{4}{11}$ to 59 generally takes a rational between 0 and 1 to an odd number (and a rational larger than 1 to an even number). Consider the function f:

$$\text{If } \; n = \sum_{k=0}^{r} e_k 2^k, e_k \in \{0,1\} \text{ then } f(n) := 2 - \sum_{k=0}^{r} e_k 2^{-k}.$$

For example, $f(59) = \frac{9}{32}$ and, in general, f maps the odd integers onto the dyadic rationals between 0 and 1. The composition $f(\omega(x))$ then takes the rationals in (0,1) to the dyadic rationals in (0,1) and coincides with Minkowski's Question Mark Function (see [W]):

$$?(x) := f(\omega(x)) = \sum_k \frac{(-1)^{k-1}}{2^{(a_1+\ldots+a_k)-1}} \text{ where } x = [0; a_1, a_2, \ldots].$$

The function $?(x)$ is purely singular, strictly increasing, and maps quadratic irrationals to rationals.

3. The (3,1) case.

We now consider

$$a_n := \sum_{3i+j=n} \binom{i+j}{i}_2 = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n-2i}{i}_2.$$

The first few terms of the sequence are:

1, 1, 1, 2, 1, 2, 2, 3, 1, 3, 2, 3, 2, 4, 3, 5, 1, 4, 3, 4, 2, 5, 3, 5, 2, 5, 4, 6, 3, 7, 5, 8, 1, 6, 4, 5, 3, 7, 4, ...

Some apparent patterns (which we shall address later) are:

$$a(2^n) = 1, a(2^n - 1) = F_{n+1}, \text{ and } a(2n) = a(n).$$

We shall derive a recursive definition of (a_n) by finding the generating function for the sequence; we develop this technique since it easily generalizes. First, we define analogs of the polynomials $(x+1)^n$ in the 'modulo 2' case: let

$$p_m := \sum_{i=0}^{m} \binom{m}{i}_2 x^i.$$

For example, $p_0(x) = 1, p_1(x) = 1 + x, p_2(x) = 1 + x^2, p_3(x) = 1 + x + x^2 + x^3$, and $p_4(x) = 1 + x^4$.

Lemma 3.1. *For $m \geq 0$,*

$$p_{2m}(x) = p_m(x^2)$$
$$p_{2m+1}(x) = (1+x)p_m(x^2).$$

Proof. If i and j are odd, then the binary expansions of i and j share a one in the last place and so, by Lemma 2.1,

$$\binom{i+j}{i}_2 = 0.$$

For any i and j, $2i$ and $2j$ share a one in the same place in their binary representations if and only if i and j do and so

$$\binom{2i+2j}{2i}_2 = \binom{i+j}{i}_2.$$

It follows that

$$p_{2m}(x) = \sum_i \binom{2m}{i}_2 x^i = \sum_i \binom{2m}{2i}_2 x^{2i} = \sum_i \binom{m}{i}_2 x^{2i} = p_m(x^2).$$

Since either i or $i-1$ is odd, at least one of $\binom{2m}{i}_2$ and $\binom{2m}{i-1}_2$ is zero and

$$\binom{2m+1}{i}_2 = \binom{2m}{i}_2 + \binom{2m}{i-1}_2.$$

Then

$$\begin{aligned} p_{2m+1}(x) &= \sum_{i=0}^{2m+1} \binom{2m+1}{i}_2 x^i = \sum_{i=0}^{2m} \binom{2m}{i}_2 x^i + \sum_{i=1}^{2m+1} \binom{2m}{i-1}_2 x^i \\ &= \sum_{i=0}^{2m} \binom{2m}{i}_2 x^i + \sum_{i=0}^{2m} \binom{2m}{i}_2 x^{i+1} = (1+x)p_{2m}(x). \quad \square \end{aligned}$$

From this, we can get a product formula and functional equation for the generating function. As a corollary, we will derive a recursive definition for (a_n). Let

$$A(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Theorem 3.2. *The generating function $A(x)$ satisfies:*

$$A(x) = (1+x+x^3)A(x^2) = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{3\cdot 2^n}).$$

Proof. We first write $A(x)$ in terms of the polynomials $p_m(x)$:

$$\begin{aligned} A(x) &= \sum_n a_n x^n = \sum_n \sum_{3i+j=n} \binom{i+j}{i}_2 x^{3i+j} \\ &= \sum_m \sum_{i+j=m} \binom{i+j}{i}_2 x^{3i+j} = \sum_m x^m \sum_{i=0}^{m} \binom{m}{i}_2 x^{2i} = \sum_m x^m p_m(x^2). \end{aligned}$$

Summing the last series over evens and odds separately and using Lemma 3.1,

$$\begin{aligned} A(x) &= \sum_{m=0}^{\infty} x^{2m} p_{2m}(x^2) + \sum_{m=0}^{\infty} x^{2m+1} p_{2m+1}(x^2) \\ &= \sum_{m=0}^{\infty} x^{2m} p_m(x^4) + \sum_{m=0}^{\infty} x^{2m+1}(1+x^2) p_m(x^4) \\ &= (1+x+x^3) \sum_{m=0}^{\infty} x^{2m} p_m(x^4) = (1+x+x^3)A(x^2). \end{aligned}$$

Iterating this formula as in the proof of Corollary 2.4, the product formula follows. □

We remark that for all three sequences studied thus far, the generating functions satisfy a rational relation among finitely many of $F(x), F(x^2), F(x^4), F(x^8), \ldots$. For example, the generating function for Gould's sequence satisfies $G(x) = (1+2x)G(x^2)$. If we let $G_n := G(x^{2^n})$, then one of the relations between G_0, G_1, and G_2 is:

$$G_0^3 G_2 - 2(G_0 + G_2)G_1^3 + 3G_0 G_1^2 G_2 = 0.$$

Sequences with such generating functions have been called "Divide and Conquer Sequences" and have been studied in [St].

The recurrence relations for (a_n) follow.

Corollary 3.3. *The sequence (a_n) may be defined recursively by: $a_0 = a_1 = 1$ and*

$$\begin{aligned} a_{2n} &= a_n \\ a_{2n+1} &= a_n + a_{n-1}. \end{aligned}$$

Proof. By Theorem 3.2,

$$\begin{aligned} &\sum a_{2n} x^{2n} + \sum a_{2n+1} x^{2n+1} = A(x) = (1+x+x^3)A(x^2) \\ &\quad = \sum a_n x^{2n} + \sum a_n x^{2n+1} + \sum a_n x^{2n+3} = \sum a_n x^{2n} + \sum (a_n + a_{n-1}) x^{2n+1}. \end{aligned}$$

Equating coefficients gives the result. □

As for the previous two sequences, (a_n) has a combinatorial interpretation. A "(3,1)-hyperbinary" representation of n is a vector $(e_k, e_{k-1}, \ldots, e_1, e_0)$ with entries $e_i \in \{0, 1, 3\}$ (the number k is not fixed) such that $\sum e_i 2^i = n$. We let $H_{3,1}(n)$ denote the set of hyperbinary representations of n. For example, $(1,3,0,3,1) \in H_{3,1}(47)$. Further, we may write

$$(1,3,0,3,1) = 3 \cdot (1,0,1,0) + (1,0,0,0,1)$$

where, of course, (1,0,1,0) and (1,0,0,0,1) are ordinary binary representations of 10 and 17 respectively. Thus

$$3 \cdot 10 + 17 = 47$$

and, furthermore, since the locations of 3's and 1's cannot coincide,

$$\binom{10+17}{10}_2 = 1$$

by Lemma 2.1. In general, it is clear that there is a one-to-one correspondence between elements of $H_{3,1}(n)$ and pairs of numbers (i, j) such that $3i + j = n$ and $\binom{i+j}{i}_2 = 1$. Hence:

Theorem 3.4. *$a_n = |H_{3,1}(n)|$ where $H_{3,1}(n)$ is the set of vectors $(e_k, e_{k-1}, ..., e_1, e_0)$ with entries $e_i \in \{0, 1, 3\}$ (the number k is not fixed) such that $\sum e_i 2^i = n$.*

The recurrence relations for (a_n) (Corollary 3.3) may now be derived using Theorem 3.4 and the method of proof of Corollary 2.7 (we leave it as an exercise for the reader).

Starting with a_1 and arranging the sequence in an array:

1																		
1	2																	
1	2	2	3															
1	3	2	3	2	4	3	5											
1	4	3	4	2	5	3	5	2	5	4	6	3	7	5	8			
1	6	4	5	3	7	4	7	2	6	5	7	3	8	5	8	.	.	.
1	9	6	7	4	10	5	9	3	8	7	10	4	11	7	11	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

There are many apparent patterns here:

$$a(2^n) = 1.$$

This is, of course, an obvious consequence of $a_{2n} = a_n$.

$$a(2^n - 1) = F_{n+1}.$$

This follows from the recurrence relations:

$$a(2^{n+1} - 1) = a(2 \cdot (2^n - 1) + 1) = a(2^n - 1) + a(2^n - 2) = a(2^n - 1) + a(2^{n-1} - 1).$$

We therefore have $a(2^n + 1) = 1 + F_n$, since $a(2^{n-1} - 1) = F_n$ implies

$$a(2^n + 1) = a(2 \cdot 2^{n-1} + 1) = a(2^{n-1} - 1) = 1 + F_n.$$

The consecutive row sums (s_n) give the sequence $1, 3, 8, 23, 67, ...$ It turns out that the Fibonacci numbers appear here:

$$s_{n+1} = 3s_n - F_n.$$

This follows from the fact that the even indexed elements of the (n+1)-st row sum to s_n and those of the odd indexed elements sum to s_n plus s_n minus the difference between the last elements of the (n+1)-st n-th rows (i.e., F_n).

The column sums involve Fibonacci numbers in that the differences are often Fibonacci numbers (e.g., the second column 2,2,3,4,6,9,... has difference sequence 0,1,1,2,3,..., the Fibonacci numbers. It is not always the case, however. But what is true is that the difference sequence is a "Gibonacci" sequence (i.e., a sequence obeying the recursive part of the definition of the Fibonacci numbers):

Proposition 3.5. *For $k \geq 0$ and any j, $0 \leq j \leq 2^k$, we have*

$$a(2^{k+2}+j) = a(2^{k+1}+j) + a(2^k+j) - a(j).$$

Proof. Assume the result true for some $k \geq 0$ and let j be less than 2^{k+1}. If j is even, say $j = 2i$, then $i \leq 2^k$ and

$$\begin{aligned} a(2^{k+3}+j) &= a(2^{k+2}+i) = a(2^{k+1}+i) + a(2^k+i) - a(i) \\ &= a(2^{k+2}+j) + a(2^{k+1}+j) - a(j). \end{aligned}$$

Assume now j odd and write $j = 2i+1$. If $i = 0$, then $a(2^{k+3}+1) = a(2^{k+2}+1) + a(2^{k+1}+1) - a(1)$ since $a(2^n+1) = 1 + F_n$. If $i \neq 0$, then $1 \leq i < 2^k$ and

$$\begin{aligned} a(2^{k+3}+j) &= a(2^{k+2}+i) + a(2^{k+2}+i-1) \\ &= a(2^{k+1}+i) + a(2^k+i) - a(i) \\ &\quad + a(2^{k+1}+i-1) + a(2^k+i-1) - a(i-1) \\ &= a(2(2^{k+1}+i)+1) + a(2(2^k+i)+1) - a(2i+1) \\ &= a(2^{k+2}+2i+1) + a(2^{k+1}+2i+1) - a(2i+1). \end{aligned}$$

By induction on k, the result follows. □

The sequence of fractions is not as well-behaved as for Stern's sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}, \frac{3}{2}, \frac{1}{3}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{2}{3}, \frac{4}{2}, \frac{3}{4}, \frac{5}{3}, \ldots$$

Obviously, the fractions are not always in lowest terms and numbers are represented several times by different fractions. An interesting question, though, is whether all positive rationals are represented.

We are not able to answer that question but it is interesting to try to follow the technique used in our proof of Theorem 2.9.

Define matrices

$$A_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } A_1 := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $v_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_1 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and, for $n \geq 2$, let $v_n := \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix}$. Then, by Corollary 3.3,

$$\begin{aligned} A_0 v_n &= v_{2n} \\ A_1 v_n &= v_{2n+1}. \end{aligned}$$

It follows that

$$\text{If } n = \sum_{k=0}^{r} e_k 2^k \ (e_k \in \{0,1\}), \text{ then } v_n = A_{e_0} A_{e_1} \cdots A_{e^r} v_0.$$

4. Some generalities.

Consider now the general case: for a and b relatively prime, let

$$c_n := \sum_{ai+bj=n} \binom{i+j}{i}_2.$$

The following facts are easy to prove using the techniques of the previous sections (we leave the proofs as exercises for the reader).

Theorem 4.1. *The following hold for the sequence (c_n):*

If $H_{a,b}(n)$ is the set of vectors $(..., e_k, e_{k-1}, ..., e_0)$ such that $\sum_k e_k 2^k = n$ and $e_k \in \{0, a, b\}$ for all k, then

$$c_n = |H_{a,b}(n)|.$$

The generating function $C(x) := \sum_{n=0}^{\infty} c_n x^n$ satisfies the functional equation

$$C(x) = (1 + x^a + x^b)C(x^2).$$

If a and b are both odd (define $a = 2a' + 1$ and $b = 2b' + 1$), then

$$c_{2n} = c_n$$
$$c_{2n+1} = c_{n-a'} + c_{n-b'}.$$

If a is even and b is odd (define $a = 2a'$ and $b = 2b' + 1$), then

$$c_{2n} = c_n + c_{n-a'}$$
$$c_{2n+1} = c_{n-b'}.$$

When placed in 'array form', the Fibonacci numbers often appear. For example, in the (3,2)-case:

0																		
1	1																	
1	0	2	1															
2	1	1	1	2	0	3	2											
3	1	3	2	2	1	2	1	3	1	2	2	3	0	5	3			
5	2	4	3	4	1	5	3	4	2	3	2	3	1	3	2	4	. . .	
8	3	7	5	6	2	7	4	7	3	5	4	6	1	8	5	7	. . .	
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .	

Here, every column is a 'Gibonacci' sequence:

$$f(2^{k+1} + j) = f(2^k + j) + f(2^{k-1} + j) \text{ if } 2^{k-1} > j.$$

The proof of this fact follows by induction and the facts that $c_{2n+1} = c_{n-1}$ and $c_{2n} = c_{2n+1} + c_{2n+3}$. Also, the row sums satisfy

$$s_{n+1} = 3s_n - 2F_{n-2} \text{ for } n \geq 2.$$

(The proof of this fact is that the recurrence relations for (c_n) give $s_{n+1} = 3s_n - 2(c(2^{n+1}-1) - c(2^n-1))$ and the fact - provable by induction - that $c(2^n-1) = F_{n-1}$.)

Another area of generalization is to consider Pascal's triangle modulo, say, 3. In this case, it is easier and perhaps more natural to deal with an alternative: let

$$C_3(n,k) := \begin{cases} 0 & \text{if } 3 \mid \binom{n}{k}, \\ 1 & \text{if } 3 \nmid \binom{n}{k}. \end{cases}$$

By Lemma 2.1, $C_3(i+j,i) = 1$ if and only if there are no carries when i and j are added in base 3. The polynomials

$$q_m(x) := \sum_{i=0}^{m} C_3(m,i)x^i$$

satisfy, as in Lemma 3.1, a set of recurrences:

$$\begin{aligned} q_{3n}(x) &= q_n(x^3), \\ q_{3n+1}(x) &= (1+x)q_n(x^3), \text{ and} \\ q_{3n+2}(x) &= (1+x+x^2)q_n(x^3). \end{aligned}$$

The generating function $T(x)$ for the sequence

$$t_n := \sum_{ai+bj=n} C_3(i+j,i)$$

satisfies the functional equation

$$T(x) = (1+x^a+x^b+x^{2a}+x^{a+b}+x^{2b})T(x^3).$$

As an example, let $a = b = 1$, and $u_n := \sum_{i+j=n} C_3(i+j,i)$. The corresponding generating function satisfies

$$U(x) = (1+2x+3x^2)U(x^3)$$

and we may derive the recurrence relations for (u_n):

$$\begin{aligned} u_{3n} &= u_n \\ u_{3n+1} &= 2u_n \\ u_{3n+2} &= 3u_n. \end{aligned}$$

The first few terms of the sequence are:

$$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, 12, 18, 2, ...$$

This is Sloane's sequence A006047; see [Sl] for details.

As another example, let $a = 2$, $b = 1$, and $v_n := \sum_{2i+j=n} C_3(i+j,i)$. The corresponding generating function satisfies

$$V(x) = (1+x+2x^2+x^3+x^4)V(x^3)$$

and we may derive the recurrence relations for (v_n):

$$\begin{aligned} v_{3n} &= v_n + v_{n-1} \\ v_{3n+1} &= v_n + v_{n-1} \\ v_{3n+2} &= 2v_n. \end{aligned}$$

The first few terms of the sequence are:

$$1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 6, 6, 6, 6, 7, 7, 8, 8, 8, , ...$$

This sequence has been studied before; see [Sl: A061392]. In that reference, the formula

$$v(n) = v(\lfloor (n+1)/3 \rfloor) + v(\lceil (n+1)/3 \rceil)$$

is given. Also there, a combinatorial interpretation is given: v_n is the number of nonnegative integers at most n having no one in their ternary representation.

A third example: let $a = 3$, $b = 1$, and $w_n := \sum_{3i+j=n} C_3(i+j, i)$. The first few terms are:

$$1, 1, 1, 2, 2, 1, 3, 2, 1, 4, 3, 2, 1, 4, 3, 2, 5, 4, 2, 5, 3, 1, 6, 4, 3, 6, 5, 2, 6, 3, 1, 7, ...$$

Its generating function is

$$W(x) = (1 + x + x^2 + x^3 + x^4 + x^6)W(x^3)$$

and the recurrence relations

$$\begin{aligned} w_{3n} &= w_n + w_{n-1} + w_{n-2} \\ w_{3n+1} &= w_n + w_{n-1} \\ w_{3n+2} &= w_n. \end{aligned}$$

We finish with a conjecture. The fact that

$$\max\{b_k : 2^n \le k < 2^{n+1}\} = b(1 + (2^{n+2} - (-1)^{n+2})/3) = F_{n+2}$$

shows that

$$\limsup_{n\to\infty} \frac{b(n)}{n^{\log_2(\phi)}} \ge 0.9588.$$

Finch [F; p.149] asks whether the following is true:

$$\limsup_{n\to\infty} \frac{b(n)}{\phi^{\log_2 n}} = 1.$$

It's irresistible to propose the following.

Conjecture.

$$\limsup_{n\to\infty} \quad \frac{1}{\gamma^{\log_2 n}} \sum_{ai+bj=n} \binom{i+j}{i}_2 = 1$$

where $\gamma > 0$ and $\gamma^{a+b} = \gamma^a + \gamma^b$.

REFERENCES

[CW] N. Calkin and H.S. Wilf, *Recounting the Rationals*, Amer. Math. Monthly, 107 no. 4 (2000) pp. 360-363.

[F] Steven Finch, "Mathematical Constants", Encyclopedia of Mathematics and its Applications #94, Cambridge Univ. Press, 2003.

[GKP] Ronald Graham, Donald Knuth, Orin Patashnik, "Concrete Mathematics", Second ed., Addison-Wesley, New York, 1994.

[Gra] Andrew Granville, *Zaphod Beeblebrox's brain and the fifty-ninth row of Pascal's triangle*, Amer. Math. Monthly, 99 no. 4 (1992) pp. 318-331.

[Gre] Thomas Green, *Recurrent Sequences and Pascal's Triangle*, Mathematics Magazine, Vol. 41, No. 1, (1968) pp. 13-21.

[R] Joseph A. Raab, *A Generalization of the Connection between the Fibonacci Sequence and Pascal's Triangle*, Fibonacci Quarterly, Vol. 1.3 (1963) pp. 21-31.

[Sc] Manfred Schroeder, "Fractals, Chaos, Power Laws", W.H. Freeman and Co., New York, 1991.

[Se] C. Series, *The modular surface and continued fractions*, J. London Math. Soc. (2) 31 (1985) no. 1, pp. 69-80.

[Sl] N.J.A. Sloane, "The On-Line Encyclopedia of Integer Sequences",
http://www.research.att.com/ njas/sequences

[St] R. Stephan, *Divide-and-Conquer generating functions. I. Elementary Sequences*, preprint, arXiv.org¿ math.CO/0307027.

[W] E. Weisstein, "Minkowski's Question Mark Function." From *MathWorld*– A Wolfram Web Resource.
http://mathworld.wolfram.com/MinkowskisQuestionMarkFunction.html

E-mail address : samuel.northshield@plattsburgh.edu