Spring 2014

A New Parameterization of Ford Circles

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November 12, 2013

Abstract

Lester Ford introduced Ford Circles in 1938 in order to geometrically understand the approximation of an irrational number by rational numbers. We shall construct Ford circles by a recursive geometric procedure and by a (well-known) parameterization by rational numbers. We introduce a new parameterization in terms of relatively prime integer solutions of $(a+b+c)^2 = a^2 + b^2 + c^2$.

1 Introduction

A Ford circle is a member of a set of circles in the plane, above and tangent to the x-axis. Starting with two circles tangent to the x-axis at (0,0) and (1,0), recursively add new circles that are tangent to the x-axis and to a pair of two larger circles; see Figure 1.

Ford circles were introduced in 1938 by Lester Ford to provide a geometric way to look at certain aspects of number theory. In particular, they provide a way to *see* Farey series [**?**] and "good" rational approximations to a given irrational number [**?**]; an approximation a/b to t is "good" if $|t - a/b| < 1/2b^2$. These good approximations are, generally, continued fraction convergents and so it it possible to understand continued fractions in terms of Ford circles (see [**?**]).

It turns out that Ford circles can be (and usually are) parameterized by rational numbers: given a rational number a/b where a, b are in lowest terms, form a circle touching $(a/b, 0)$ with radius $1/2b^2$. That is, every Ford circle (defined geometrically) is of the form $C_{a,b}$ for some rational a/b and, conversely, every $C_{a,b}$ arises from the geometric construction. We shall prove this below.

We also introduce a new way to parameterize Ford circles in terms of relatively prime integer solutions of the equation $a^2 + b^2 + c^2 = (a+b+c)^2$. We use this to show the surprising fact that every relatively prime integer solution (a, b, c) of this equation satisfies the sum of any two is, up to sign, a perfect square.

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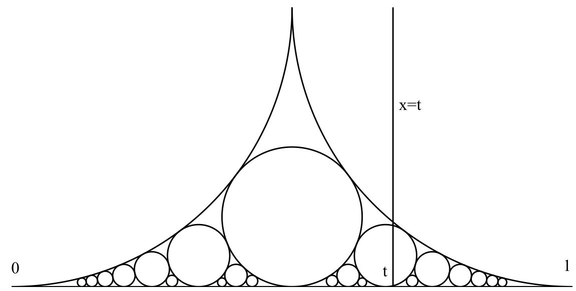

Figure 1: Ford circles with a line through a number t

This paper arose out of a research project that the first author completed as a student with guidance by the second author. We include, in the last section, a sketch of a higher dimensional analogue by the second author.

2 An interesting equation

Consider the equation

$$(a+b+c)^2 = a^2 + b^2 + c^2. \qquad (1)$$

It is *homogeneous*: if (a, b, c) is a solution, then so is (ka, kb, kc) for any real number k. It is then clear that any integer solution is a constant times a relatively prime integer solution (i.e., the three numbers share no common prime divisor).

Here are some relatively prime integer solutions:

$$(2, 2, -1), (-2, 3, 6), (-4, -12, 3), (30, 70, -21), (-28, 12, -21)$$

We observe that the sum of any two elements of a given solution satisfies $|a+b| = n^2$. We shall investigate this property more fully; first we give a direct proof and later explain how Ford circles help with another proof.

One way to see that $|a + b|$ is a perfect square is to first look at all possible solutions. Given a, b and solving (1) for c gives $c = -ab/(a + b)$. Hence $(a, b, -ab/(a + b))$ is a solution of (1) for *any* a, b. By multiplying by an appropriate k, we can get a relatively prime integer solution. For example,

$1, 2 \mapsto (1, 2, -2/3) \mapsto (3, 6, -2)$
$12, 24 \mapsto (12, 24, -8) \mapsto (3, 6, -2)$
$3, 6 \mapsto (3, 6, -2) \mapsto (3, 6, -2).$

The last line indicates that every relatively prime integer solution arises from this process. Without loss of generality, assume m and n are relatively prime. Then

$$m, n \mapsto (m, n, -mn/(m+n)) \mapsto (m(m+n), n(m+n), -mn). \qquad (2)$$

This shows that every relatively prime integer solution is of the form $(m(m+n), n(m+n), -mn)$ and therefore the absolute value of the sum of any two elements is one of $(m+n)^2, m^2, n^2$, a perfect square.

Is this property a specific case of something more general or is it, in some sense, just a coincidence? What we shall do is show that it follows from two different ways of looking at "Ford circles".

3 Getting to know Ford circles

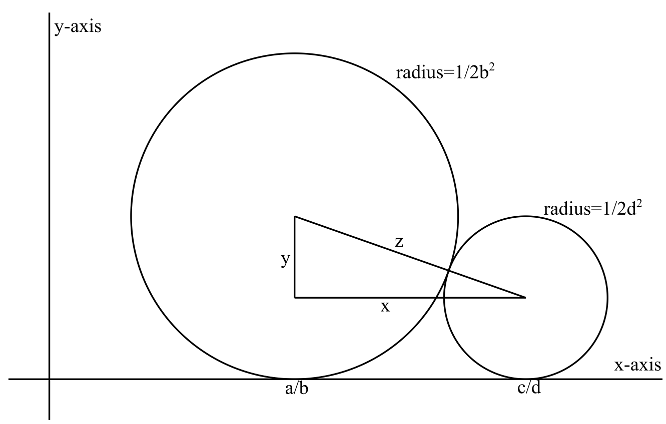

Figure 2: Two tangent Ford circles.

A Ford circle, denoted $C_{a,b}$, is tangent to and above the x-axis at a point $(a/b, 0)$ with radius $1/2b^2$ where a and b are relatively prime integers. We now look at when two Ford circles are tangent to one another. Looking at Figure 2, we see the two circles are tangent if $x^2 + y^2 = z^2$ (by the Pythagorean theorem) or, equivalently, if

$$\left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2. \tag{3}$$

For convenience, we write $C_{a,b}||C_{c,d}$ if the two Ford circles $C_{a,b}$ and $C_{c,d}$ are tangent. Expanding and then cancelling, we see that (3) is equivalent to $(ad - bc)/b^2d^2 - 1/2b^2d^2 = 1/2b^2d^2$ and thus

$$C_{a,b}||C_{c,d} \text{ if and only if } |ad - bc| = 1. \tag{4}$$

Since $|ad - bc|$ is a non-negative integer, if $|ad - bc| < 1$, then $|ad - bc| = 0$ and so $a/b = c/d$ and, by relative primality, $|a| = |c|, |b| = |d|$ and the circles $C_{a,b}$ and $C_{c,d}$ are identical. If $|ad - bc| > 1$ then, by (3), the distance between the centers of the two circles is larger than the sum of their radii: the circles are disjoint.

Given two tangent circles $C_{a,b}, C_{c,d}$ it is clear that there is a third circle tangent to each; by (4), it is easy to verify

$$C_{a,b}||C_{a+c,b+d}||C_{c,d} \text{ and } C_{a,b}||C_{a-c,b-d}||C_{c,d}.$$

This gives rise to "mediant addition", often joked about as "elementary school addition of fractions": $a/b \oplus c/d = (a + c)/(b + d)$. The x-values of the tangent points of two tangent Ford circles "add" to give the x-value of the tangent point of the circle tangent to both.

Ford circles are intimately connected with number theory. Among the most basic is in Diophantine approximation [**?**], [**?**] but there is also the famous "circle method" [**?**] in analytic number theory and the close connection with continued fractions [**?**]. There is even a connection with the Riemann hypothesis [**?**]

We here illustrate the connection with Diophantine approximation. In Figure 1 we see several Ford circles, one of which is crossed by a vertical line through an irrational number t. That line necessarily crosses infinitely many Ford circles and, for each such circle $C_{a,b}$, Figure 3 shows that

$$\left|t - \frac{a}{b}\right| < \frac{1}{2b^2}.$$

Hence, we find that for any irrational t, there are infinitely many integer pairs a, b such that $|t - a/b| < 1/2b^2$. This is stronger than the well-known theorem that $|t - a/b| < 1/b^2$ for infinitely many pairs a, b (Th. 185 of [**?**]) but is weaker than the fact that $|t - a/b| < 1/\sqrt{5}b^2$ for infinitely many pairs a, b (and $\sqrt{5}$ is the largest number for which this is true – see Theorems 193 and 194 of [**?**]).

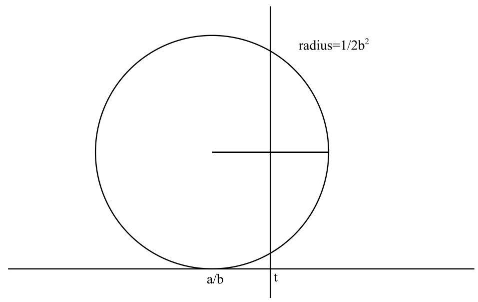

Figure 3: How a line crosses a Ford circle

4 A geometric recursive construction of Ford circles

The geometric construction will be to start with a collection of just two circles above and tangent to the x-axis at (0,0) and (1,0) respectively, each with radius 1/2. The recursive step is, given any two tangent circles in the collection generated thus far, add the unique third one tangent to each of the given two circles. Repeat *ad infinitum.*

For this procedure to even make sense, we need to know:

- Given any two tangent Ford circles, there is another one tangent to each of those with tangent point on the x-axis in between,
- That circle is unique.

Given any $x \in \mathbb{R}$ and any $r > 0$, the circle tangent to the x-axis at $(x, 0)$ with radius r may be written as $C_{x/\sqrt{2r},1/\sqrt{2r}}$. Hence every circle above and tangent to the x-axis is of the form $C_{a,b}$ for some *real* a, b.

Given any two tangent circles $C_{a,b}, C_{c,d}$ that are tangent to each other, we have seen that $|ad - bc| = 1$ and $C_{a+c,b+d}$ is tangent to each.

Suppose now that $C_{u,v}$ is another circle tangent to both. By (3), we have the following matrix equation:

$$\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}$$

which, since the square matrix has determinant ± 1, has solution $u = \pm a \pm c$, and $v = \pm b \pm d$. Hence we have, by the condition that u/v is between a/b and

c/d, that $C_{u,v} = C_{a+c,b+d}$. We have thus shown that the geometric procedure makes sense.

In terms of Ford circle notation, we start with $C_{0,1}$ and $C_{1,1}$. It is then clear that every step of the geometric procedure adds a circle of the form $C_{a,b}$ where a and b are integers. We claim that every $C_{a,b}$ in the geometric construction has $\gcd(a,b)$. Suppose not, then there are two pairs $\gcd(a,b)$ and $\gcd(c,d)$ with $|ad - bc| = 1$ but for which $a + c$ and $b + d$ are divisible by some prime p. But then $|(a + c)d - (b + d)c| = 1$ is divisible by p also – a contradiction. Therefore, every circle arising from the geometric construction is a Ford circle.

We next show that every Ford circle arises this way and therefore the following theorem holds.

Theorem 1 *The circles formed in the geometric recursive procedure above are precisely the Ford circles.*

PROOF: We will show, by induction, that every Ford circle is in the geometric construction. Let

$$\mathbf{G} = \{(a,b) : C_{a,b} \text{ arises from the geometric recursive construction.}\}.$$

Let P_n be the proposition "$(j,k) \in \mathbf{G}$ whenever $j/k \in [0,1]$, $\gcd(j,k)$, and $k < n$".

P_2 is true since $(0,1)$ and $(1,1)$ are in $\mathbf{G}$. Now suppose P_n is true. To show that P_{n+1} is true, it is enough to show that a Ford circle $C_{a,n}$ is tangent to two Ford circles $C_{u,v}, C_{a-u,n-v}$ which are tangent to each other and for which $v < n$ and $n - v < n$. This will happen if we can find u, v satisfying

$$1 \leq v < n \text{ and } |av - nu| = 1.$$

We know $\gcd(a,n)$. According to Saracino [**?**], there exist natural numbers x, y such that $ax+ny = 1$. Since there is a unique integer k such that $0 \leq kn+x < n$, then letting $u = ka - y$ and $v = kn + x$ defines u, v as desired.

By the principle of strong mathematical induction, the theorem is shown. QED

5 A new parameterization of Ford circles

We define, for real numbers s and t, $\langle s,t\rangle$ to be the circle above and tangent to the x-axis at $t/(s + t)$ with radius $1/2(s + t)$; see Figure 4.

It is easy to check that for any real x and positive r, $\langle(1 - x)/2r, x/2r\rangle$ is the circle tangent to the x-axis at $(x, 0)$ with radius r. Therefore, every circle above and tangent to the x-axis is (uniquely) representable as the circle $\langle s,t\rangle$ for some real s and t.

Lemma 1 *For any m, n, $C_{n,m+n} = \langle m(m + n), n(m + n)\rangle$.*

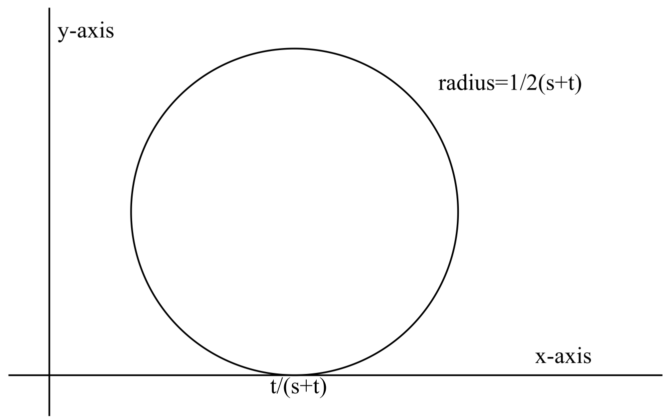

Figure 4: The circle $\langle s,t\rangle$

PROOF: $C_{n,m+n}$ is the circle tangent to the x-axis at $(n/(m+n),0)$ with radius $1/2(m+n)^2$. So is $\langle m(m+n), n(m+n)\rangle$. QED

Given a Ford circle $C_{a,b}$, let $m := b-a$ and $n := a$. Since a and b are relatively prime, so are m and n. Then, by Lemma 1, $C_{a,b} = C_{n,m+n} = \langle m(m+n), n(m+n)\rangle$.

Recall formula (2) which described a way of representing *every* relatively prime integer solution of (1) in terms of relatively prime m,n:

$$m,n \mapsto (m(m+n), n(m+n), -mn).$$

Hence $C_{a,b} = \langle s,t\rangle$ for some relatively prime integer solution (s,t,u) of (1). It is easy to reverse this analysis and conclude

Theorem 2 *The set of Ford circles is*

$$\{\langle s,t\rangle : s^2+t^2+u^2 = (s+t+u)^2,\ s,t,u \text{ are relatively prime, } s+t>0\}.$$

The technical detail $s+t>0$ is necessary to rule out negative radii.

Corollary 1 *If a,b,c are relatively prime integers satisfying $a^2+b^2+c^2 = (a+b+c)^2$, then $|a+b|$ is a perfect square.*

PROOF: The set of radii of Ford circles comprises the set $\{1/2b^2 : b \text{ an integer }\}$ but also comprises the set

$$\{1/2(a+b) : a^2+b^2+c^2=(a+b+c)^2,\ a,b,c \text{ are relatively prime, } a+b>0\}.$$

QED

6 3rd dimension

In this section we indicate one direction how this generalizes up one dimension. This is current work by the second author.

We fix three points P_0, P_1, P_2 in $\mathbb{C}$ which form an equilateral triangle of side length 1 (e.g., let $P_0 = 0$, $P_1 = 1$, and $P_2 = 1+\omega$ where ω is the cube root of unity $(-1+\sqrt{-3})/2$). Given three real numbers a, b, c, we form a sphere as indicated in Figure 5.

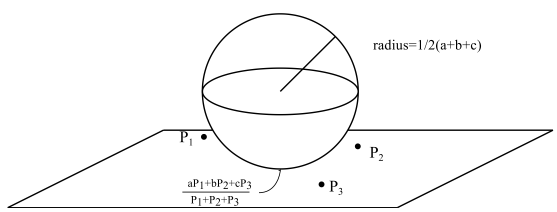

Figure 5: The sphere $\langle a, b, c\rangle$

It turns out that every sphere in $\mathbb{C}\times[0,\infty)$ tangent to $\mathbb{C}$ is uniquely represented by $\langle a, b, c\rangle$ for some real numbers a, b, c.

A reasonable generalization of Ford circles is the collection of Ford spheres defined as

$$\{\langle a,b,c\rangle : a^2+b^2+c^2+d^2=(a+b+c+d)^2,\ a,b,c,d \text{ are relatively prime, } a+b+c>0\}.$$

Indeed, it turns out that these spheres have disjoint interiors and each such sphere is tangent to infinitely many others.

If we adjoin ω to the integers, then we get the ring $\mathbb{Z}[\omega] := \{a+b\omega : a, b \in \mathbb{Z}\}$. It turns out to be a unique factorization domain and so the concept of "relatively prime" makes sense. If we let $S_{\alpha,\beta}$ be the sphere in $\mathbb{C} \times [0, \infty)$ tangent to $\mathbb{C}$ at α/β with radius $1/2|\beta|^2$ then the set of spheres

$$\{S_{\alpha,\beta} : \alpha, \beta \text{ relatively prime in } \mathbb{Z}[\omega]\}$$

coincides with the set of Ford spheres.

Analogously to Corollary 1, every relatively prime integer solution of $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2$ has $|a + b + c| = |m + n\omega|^2 = m^2 - mn + n^2$ for some integers m, n. We note that solves a recent problem appearing in the American Mathematical Monthly [**?**]

Starting with the collection of three spheres $\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle$, we perform the following recursive step *ad infinitum*: Given any three mutually tangent spheres in the collection generated thus far, add both (there are two) spheres that are tangent to each of the given three spheres and to the x, y-plane. The limiting collection is, again, the set of Ford spheres.

References

[1] Burger, E., "Exploring the number jungle: a journey into Diophantine analysis", Student Mathematical Library, **8**, AMS, Providence, RI, 2000.

[2] Conway, J.H., "The Sensual (Quadratic) Form", Carus Mathematical Monographs, **26**. MAA, Washington D.C., 1997.

[3] Hardy, G.H., Wright, E.M., "An Introduction to the Theory of Numbers" (5th ed.), Oxford at the Clarendon Press, Oxford, 1979.

[4] Northshield, S., *Stern's diatomic sequence 0,1,1,2,1,3,2,3,1,4,...,* American Math. Monthly **117** (2010), no. 7, 581-598.

[5] Northshield, S., Problem no. 11671, American Math. Monthly **119** (2012) no. 9, p. 800.

[6] Rademacher, H., *On the expansion of the partition function in a series,* Ann. Math. **44**, no. 3, (1943) 416-422.

[7] Saracino, D., "Abstract Algebra, a first course", 2nd ed., Waveland Press, Long Grove, IL, 2008.

[8] Series, C., *The modular surface and continued fractions,* J. London Math. Soc. (2) **31** (1985), no. 1, 69-80.

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